Exploring many-body interactions through quantum Fisher information

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(Received 8 January 2024; accepted 7 June 2024; published 1 July 2024)

The investigation of many-body interactions holds significant importance for both quantum foundations and information. Hamiltonians coupling multiple particles at once, beyond other applications, can lead to faster entanglement generation, multiqubit gate implementation, and improved error correction. As an increasing number of quantum platforms can enable the realization of such physical settings, it has become interesting to study the verification of many-body-interaction resources. In this work, we explore the possibility of higher-order coupling detection through the quantum Fisher information. For a family of normalized symmetric k-body Ising-like Hamiltonians, we derive bounds on the quantum Fisher information in product states. Due to its ordering with respect to the order of interaction, we demonstrate the possibility of detecting many-body couplings for a given Hamiltonian from the discussed family by observing violations of an appropriate bound. As a possible extension to these observations, we further analyze an example concerning three-body-interaction detection in the XY model.

DOI: 10.1103/PhysRevA.110.012407

I. INTRODUCTION

Among many fascinating phenomena in physics, in recent years studying the nature of interactions has become not only a subject of fundamental research but also part of the current pursuit towards modern quantum technologies. Most current controllable quantum systems rely solely on the two-body interactions between particles [1–3]. Nevertheless, many-body interactions are often discussed in the context of effective models in low-energy physics [4–6]. These include studies of spin systems [7–11], extended Hubbard models describing ultracold atoms or molecules in optical lattices [12–24], quantum chemistry [25–28], and nuclear and particle physics [29–32]. Further applications of higher-order interactions can be found in entanglement generation [33–36] and error correction [37–39], among others [40–43]. Thus, the search for many-body interactions plays a significant role for both quantum foundations and future quantum technologies [44–49]. With rising demand for the implementation of many-body-interaction Hamiltonians, it has become interesting to study their verification [43]. A universal method solving this task could probe new physical effects and give insights into how to engineer the desired Hamiltonians. Furthermore, it would be useful for Hamiltonian-learning protocols because they often require prior knowledge of the maximal degree of an interaction graph (see, e.g., [50]). On the other hand, it would answer the question of whether the Hamiltonian is even worth learning if we are interested in its many-body-interaction properties. In this work, we demonstrate that the existence of genuine many-body interactions can be verified through quantum Fisher information (QFI), thus possibly paving the way for a new area of nonlocal-interaction research.

QFI has been studied in various contexts [51–61], including quantum phase transitions [62–67] and, most notably, quantum metrology [68,69]. For a given Hamiltonian it allows one to find states that guarantee measurement precision beyond the classical limit [68,69]. An experimental measurement, or estimation, of QFI can be performed via several techniques. The proposed theoretical protocols take advantage of the dynamical susceptibility [70], projections onto the initial state (Loschmidt-echo protocol) [71], overlap detection [72], randomized measurements [73–75], and adiabatic perturbation theory [76]. A direct QFI (or its lower bound) measurement was performed in, e.g., [74,77,78].

In the standard metrological scenario, Hamiltonians under consideration are strictly local. However, many-body interacting systems were also examined in terms of improved scaling [69]. Here, to make our presentation as simple as possible, we study the family of k-local permutationally invariant Ising-like Hamiltonians of N particles. For them, we illustrate the main premise behind this paper, which is that QFI can be used to detect the interaction order. Based on this observation, we derive bounds on the Hamiltonians with at most two-body-interaction terms, showing the possibility of witnessing the presence of k-body interactions with product states (see Fig. 1). As a possible extension, we also discuss an example of a similar study for the XY model.

II. MOTIVATING EXAMPLE

First, we start with a simple example that motivates our work. Consider a system of three qubits on a triangle that interact in the σ_z direction. Fixing the order of interaction at

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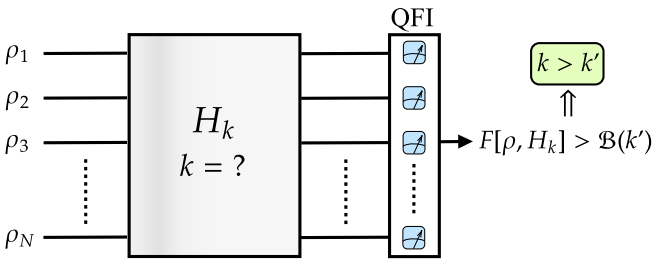

FIG. 1. Motivation. In this work, we explore the order of interactions, i.e., the maximal number of simultaneously coupled qubits in a Hamiltonian, through the use of quantum Fisher information. For symmetric Ising-like Hamiltonians H_k (7), we show that quantum Fisher information calculated in a product state is bounded with a bound dependent on k. This allows one to detect whether H_k manifests interactions beyond the k'-body case, where $k > k'$. The basic idea is as follows. Starting from an arbitrary separable state, we let the particles interact via H_k, and then we measure its quantum Fisher information. Based on our findings, we check whether the result is greater than the maximally allowed value for the chosen k'. If the answer is positive, we can claim that H_k contains terms of at least $(k'+1)$th order.

2, the two-body-interaction Hamiltonian is given as

$$H_2 = \tfrac{1}{2}(\sigma_z \otimes \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z \otimes \sigma_z + \sigma_z \otimes \mathbb{1} \otimes \sigma_z),$$

where, in order to keep the correspondence with standard metrological notation, the prefactor was chosen such that the maximal eigenvalue does not exceed $N/2 = 3/2$. On the other hand, focusing on only the three-body couplings, we get

$$H_3 = \tfrac{3}{2}(\sigma_z \otimes \sigma_z \otimes \sigma_z).$$

Now, let us consider the maximization of quantum mechanical variance $(\Delta H)^2 = \langle H^2\rangle - \langle H\rangle^2$ over the set of pure product states. For H_2 the optimal product state, i.e., the state that maximizes the variance, is given as $|\psi_{\text{prod}}\rangle = (\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle)^{\otimes 3}$, where $|0\rangle$ and $|1\rangle$ are eigenstates of σ_z and $p = (3+\sqrt{3})/6$. See the Appendix for direct calculations. The exact value of the maximal variance is $(\Delta H_2)^2_{\max} = 1$. It is straightforward to show that for trivial Hamiltonians with one-body terms the variance is given as $(\Delta H_1)^2_{\max} = N/4$. Consequently, for three qubits we get $(\Delta H_1)^2_{\max} = 3/4$, which is smaller than in the two-body case. Moving on to the higher-order-interaction Hamiltonian, the optimal state is *no longer* a tensor product of identical local states $|\psi\rangle^{\otimes 3}$. It might seem that the form of the state that maximizes the variance for the two-body interaction is trivially obtained from the symmetry, but it is not. In fact, the only condition arising from it is that $\langle\psi_1\psi_2\psi_3|H|\psi_1\psi_2\psi_3\rangle = \langle\psi_2\psi_1\psi_3|H|\psi_2\psi_1\psi_3\rangle$, with similar conditions for all other permutations of indices. This can be seen clearly in the three-body-interaction case. Examining the spectrum of H_3, we can construct a product state which is a superposition of eigenstates associated with the maximal and minimal eigenvalues (see the Appendix). Namely, we have $|\psi_{\text{prod}}\rangle = |00\rangle \otimes 1/\sqrt{2}\,(|0\rangle + |1\rangle)$ and the corresponding $(\Delta H_3)^2_{\max} = 9/4$. Limiting ourselves to the product states being a tensor product of identical one-qubit states, we would arrive at the value accessible for $k = 2$. From these results, it is apparent that

$$L_{\text{S}} = (\Delta H_1)^2_{\max} < (\Delta H_2)^2_{\max} < (\Delta H_3)^2_{\max} \leftarrow L_{\text{H}},$$

where we denote the values of variances coinciding with the standard limit and the Heisenberg limit for one-body Hamiltonians and entangled states as L_{S} and L_{H}, respectively. Here, it is also worth noting that the ordering of maximal variance is not due to the possibility of obtaining higher eigenvalues by changing the interaction type or the number of terms in the Hamiltonian since they are all normalized in the same manner. This simple example shows that variance of a Hamiltonian calculated in a product state may be exploited as a tool for detecting many-body-interaction terms.

III. PHYSICAL MODEL AND PRELIMINARIES

In the following, to make our considerations more general, instead of the variance of a Hamiltonian, we will focus on the quantum Fisher information. For any Hermitian operator A and state ρ it is defined as

$$F[\rho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{(\lambda_k + \lambda_l)} |\langle k|A|l\rangle|^2, \tag{1}$$

with $|k\rangle$ and λ_k being the eigenvectors and eigenvalues of a density matrix ρ, respectively, and the sum being evaluated only when the denominator is different from zero. Most commonly, it is used in quantum metrology because it allows one to find a suitable state that one can use to boost phase-measurement sensitivity. It was shown that for local Hamiltonians QFI in product states is bounded by N [79]. Introducing entangled states moves this bound to N^2, leading to the so-called Heisenberg limit [79,80]. Moreover, for any quantum state ρ a pure state $|\psi\rangle$ exists for which [68]

$$F[\rho, A] \leqslant 4(\Delta A)^2_{|\psi\rangle} = F[|\psi\rangle, A]. \tag{2}$$

Hence, any maximization of QFI can be performed only among the pure states. This justifies the choice of variance and calculations presented in the previous section.

Our task is to construct a model and derive a bound on the quantum Fisher information in a *product state* that is an increasing function of an interaction order. Now, let us precisely define the notion of higher-order interactions. A given Hamiltonian

$$H_k = \sum_j h_j^{(k)} \tag{3}$$

is considered to be k-local if each $h_j^{(k)}$ acts nontrivially on at most k particles. Hamiltonians which act exactly on k or at most on K particles will be denoted as H_k and $\mathcal{H}^{(K)}$, respectively. By this definition, we will use the k-locality and kth order of interactions interchangeably. Measurements enhanced with Hamiltonians that are nonlinear in operators have been studied in terms of sensing and scaling in the past [69]. They include, for example, products of photon creation and annihilation operators [81–84], angular momentum and its components [85–88], many-body models [59,60,89–92], and, more generally, powers of the sum of local operators

[93], as well as the general k-body [94] and symmetric k-body-interaction Hamiltonians [93,95], with no comparison between entangled and separable states in the latter cases. However, in this work, we study a specific scenario which resembles the standard metrological approach but will be used in a far different context. For such means, we will consider only the symmetric Ising-like Hamiltonians. First, we define the auxiliary Hamiltonians that contain only the k-body-interaction terms as

$$H_k = \mathcal{N} \sum_{(i_1,\cdots,i_k)\in G_k} \sigma_z^{i_1}\sigma_z^{i_2}\cdots\sigma_z^{i_N}, \tag{4}$$

where $\mathcal{N}$ is a normalization constant and G_k is a fully connected interaction graph for k-body interactions; the summation is performed over all k-partite subsets of particles, making it permutationally invariant. In the case of $k = 1$ we retrieve the standard metrological Hamiltonians

$$H_1 = \frac{1}{2}\sum_{i=1}^{N}\sigma_z^i \tag{5}$$

if proper normalization is chosen. Another example can be given for $k = 2$ and $\mathcal{N} = J$, namely,

$$H_2 = J\sum_{i<j}^{N}\sigma_z^i\sigma_z^j. \tag{6}$$

Note that the above Hamiltonian represents a long-range-interaction Ising model on a complete interaction graph. For more examples see Secs. II and V. Now, a general symmetric Ising-like Hamiltonian containing at most k-body interactions can be constructed as

$$\mathcal{H}^{(K)} = \mathcal{N}\sum_{k\leqslant K}\alpha_k H_k, \tag{7}$$

where, again, $\mathcal{N}$ is a normalization constant and α_k are real numbers. Throughout most of this paper, we will focus on $\forall_k\,\alpha_k = 1$ and discuss its modifications in the examples. It is worth noting that all of the results presented here hold for any Hamiltonian equivalent under local unitary operations. This follows from the property of QFI which states that $F[\rho, U^\dagger A U] = F[U\rho U^\dagger, A]$ and local unitary invariance of entanglement. Hamiltonians from a different class will be discussed in Sec. VI.

Before we move on to our results, we need to specify a proper normalization for $\mathcal{H}^{(K)}$. Our motivation is to test the order of interactions present in the system. Moreover, we would like to compare our results with the metrological approach and stay consistent with its results. In order not to break the classical and Heisenberg scaling we choose to set the operator norm $||\mathcal{H}^{(K)}|| = \max_\phi ||\mathcal{H}^{(K)}|\phi\rangle|| = N/2$. Note that we want the Hamiltonian to appear as if no k-body interactions were present in it. If we dropped this assumption, we would obtain the nonlinear Hamiltonians scalings $N^{\sim k}$ (see, e.g., [81–93]). Our approach leads to an upper bound on variance and QFI which cannot exceed N^2 for any k and quantum state $|\psi\rangle$, entangled or not. We implement this norm by setting $\mathcal{N}$ to $N/(2\max_i|E_i|)$, where E_i is an eigenvalue of $\mathcal{H}^{(K)}$. As this normalisation is multiplication by a constant, there still is an overlap between the previous research and our results. This will be commented on in Sec. V.

IV. INTERACTION-DEPENDENT BOUNDS ON QFI FOR PRODUCT STATES

For the considered model it is possible to derive the explicit formulas for the eigenvalues based on its symmetry. For the k-local Hamiltonian defined in (7) and included normalization we get

$$\Omega_e^{N,K} = \sum_{k\leqslant K}\omega_e^{N,k}, \quad \omega_e^{N,k} = \sum_{j=0}^{e}\frac{N}{2}\frac{\binom{e}{j}\binom{N-e}{k-j}}{\binom{N}{k}}(-1)^j, \tag{8}$$

where e is the number of excitations, i.e., the number of $|1\rangle$ elements in the N-qubit state. Here, $\omega_e^{N,k}$ represents the eigenvalues of a Hamiltonian with only k-body terms.

As a first step, we will limit ourselves to a scenario with a fixed k, i.e., H_k. In such a case, the maximization of variance and hence the QFI (2) over pure product states can be performed as follows. Since variance is the function of the square modulus of amplitudes and Hamiltonian eigenvalues, we can consider only product states of the following form:

$$|\psi_{\text{prod}}\rangle = \bigotimes_i^N(\sqrt{p_i}|0\rangle + \sqrt{1-p_i}|1\rangle).$$

Using this parametrization and calculating the variance, we get

$$(\Delta H_k)^2 = \sum_{l_1,\ldots,l_N=0,1}\prod_{i=1}^{N}p_i^{l_i}(1-p_i)^{1-l_i}\left(\omega_{l_1+\cdots+l_N}^{N,k}\right)^2 - \left(\sum_{l_1,\ldots,l_N=0,1}\prod_{i=1}^{N}p_i^{l_i}(1-p_i)^{1-l_i}\omega_{l_1+\cdots+l_N}^{N,k}\right)^2.$$

A necessary condition for the existence of a multivariable function extremum is the disappearance of its first derivatives. Taking derivatives of $(\Delta H)^2$ over p_i, we arrive at the system of N equations linear in p_k,

$$\begin{aligned}0 &= \frac{\partial(\Delta H_k)^2}{\partial p_k}\\ &= \sum_{l_1,\ldots,l_N=0,1}(-1)^{1-l_k}\prod_{i\neq k}^{N}p_i^{l_i}(1-p_i)^{1-l_i}\left(\omega_{l_1+\cdots+l_N}^{N,k}\right)^2\\ &\quad - 2\left(\sum_{l_1,\ldots,l_N=0,1}\prod_{i=1}^{N}p_i^{l_i}(1-p_i)^{1-l_i}\omega_{l_1+\cdots+l_N}^{N,k}\right)\\ &\quad \times\sum_{l_1,\ldots,l_N=0,1}(-1)^{1-l_k}\prod_{i\neq k}^{N}p_i^{l_i}(1-p_i)^{1-l_i}\omega_{l_1+\cdots+l_N}^{N,k},\end{aligned} \tag{9}$$

where the dependence on p_k is only in the term in the parentheses. Since we know all ω's, this can be solved directly, and the resulting set of $(p_1,\ldots,p_N)$ satisfying $0\leqslant p_i\leqslant 1$ is the solution to our optimization problem. We present a step-by-step solution for $N = 3$ in the Appendix.

V. TESTING *k*-BODY INTERACTIONS

The most interesting case for higher k-body-interaction terms would be to find an explicit bound on QFI for $k = 2$. For fixed $k = 1$ we once again refer to the fundamental result obtained for local Hamiltonians [79]. One of its many consequences is the classical scaling with $\max_{|\psi_{\text{prod}}\rangle} 4(\Delta H_1)^2 = N$. It is straightforward to see that our approach is consistent with that. With direct solutions of (9) and numerical calculations, for $k = 2$ we notice that up to $N = 13$, the solution to the given optimization problem is obtained with $|\psi_{\text{prod}}\rangle = (\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle)^{\otimes N}$. For more discussion see the Appendix. This result is known to hold asymptotically when the number of particles is much greater than the interaction order, here $n \gg 2$ [93]. It is valid in our approach because a nonlinear Hamiltonian $(\sum_i H_1)^2$ for which this result was derived is equal to $N\mathbb{1}\cdots\mathbb{1} + NH_2$ and thus holds in our case. Furthermore, it is consistent with the results obtained for the Lipkin-Meshkov-Glick (LMG) model and the nearest-neighbor (as well as fully connected) Ising model with an interaction parameter smaller than its critical value (general parameter range) [91,92]. This will also be elaborated on later. For the given state, the variance reduces to

$$(\Delta H_2)^2 = \frac{4}{N-1}[-2N(2N-3)p^4 + 4N(2N-3)p^3 - N(5N-7)p^2 + N(N-1)p]. \tag{10}$$

By finding zeros of its derivative we get three unique roots, from which the one that maximizes the variance is given as

$$p_{\max} = \frac{2N-3+\sqrt{2N^2-7N+6}}{4N-6}. \tag{11}$$

The resulting maximal QFI in a product state, i.e., the solution to (9) obtained with the above $p_{\max}$, for $k = 2$ is

$$F_{\max}[|\psi_{\text{prod}}\rangle, H_2] = \frac{2N(N-1)}{2(N-2)+1}. \tag{12}$$

The results presented here coincide with the ones obtained for the LMG model in the limit of large coupling constant γ and the context of statistical speed (see the Supporting Information of [91]). This is due to the fact that once the coupling constant is large, the single-qubit terms in the LMG model can be neglected. Then, up to a constant, the LMG Hamiltonian is equivalent to H_2 studied here. Note that (12) is an increasing function of N which bounds, from the above, the one-body scaling. Explicit results for $k \leqslant 5$ and $N \leqslant 10$ are presented in Fig. 2. As expected, for $k = 1$ the maximal QFI scales as N [79]. For $k > 1$, the results clearly show that the maximal Fisher information in a product state is ordered with respect to the fixed interaction order k. This observation motivates our goal of testing the presence of k-body interactions with QFI and product states. We should also mention that for $k = N$ the QFI in a product state is maximal, as in [89], and a constant trend for $N < 2k$ is observed (see Fig. 2).

A natural extension of the above considerations is to fix the number of qubits N and study bounds on maximal QFI with Hamiltonians H_k containing at most k-body couplings. Technically, knowing the eigenvalues of such Hamiltonians [see (8)], we can once again maximize the variance and hence

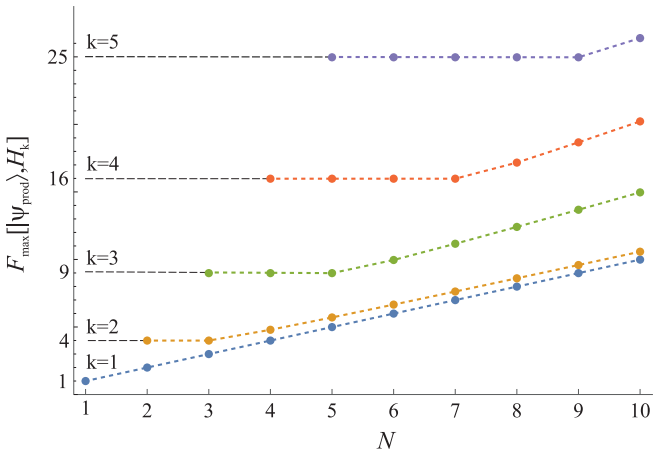

FIG. 2. Maximal quantum Fisher information of k-local Hamiltonian H_k in a product state as a function of the number of particles N. Numerical solutions of (9) for different choices of k are plotted in different colors, and dashed lines are guides to the eye. Values associated with $k = 1$ scale linearly with N [79], while for $k = 2$ their behavior is described by (12). From the plot, one can see that a clear ordering of the maximal QFI with respect to the fixed interaction order exists. This observation allows one to formulate criteria which would distinguish the minimal order of k-body interactions present in the Hamiltonian using separable states. According to our choice of normalization when $N = k$, Heisenberg scaling of N^2 is achieved as expected. Note that for $N < 2k$ the trend is constant. Then, once the number of particles is equal to $2k$ the QFI grows. This change in the behavior of QFI is necessary because it would be impossible for it to remain constant since it would eventually yield values smaller than the ones obtained with local Hamiltonians and would lead to a contradiction.

QFI directly. Here, however, we do not want to examine a sum of k-local Hamiltonians, but its behavior when the many-body couplings are varied. To detect interactions of order $k > 2$ it is always sufficient to violate the bound for $\mathcal{H}^{(2)} = H_1 + H_2$. Again, by solving (9) we can calculate the desired limits on QFI. As before, the optimal solution is found to be realized by $p_i = p$ for all i. For our problem, the exact calculations and numerical optimization for up to $N = 13$ give rise to the extrapolated pattern

$$\begin{aligned}\mathcal{B}_{1+2} &= \max_{|\psi_{\text{prod}}\rangle} F[|\psi_{\text{prod}}\rangle, \mathcal{H}^{(2)}] \\ &= \max\left(-\frac{16Np(p-1)}{(N+1)^2}\{(N-2)^2 \right. \\ &\quad \left. + 2(N-1)p[(2N-3)p-(2N-5)]\}\right)\end{aligned} \tag{13}$$

and allow us to formulate the following criterion:

$$\text{if } F[\rho, \mathcal{H}^{(K)}] > \mathcal{B}_{1+2} \Rightarrow K \geqslant 3. \tag{14}$$

The polynomial to be maximized is fourth order, and a closed formula for the maxima can be found explicitly. However, due to its extensive structure, we chose to present the result in the above form. Explicit values of $\mathcal{B}_{1+2}$ for chosen N are shown in Table. I. It is worth noting that $\mathcal{B}_{1+2}$ never exceeds the maximal QFI for only two-body interactions, and if needed, a stricter bound can be chosen. Furthermore, the presented

TABLE I. Explicit values of the maximal quantum Fisher information $\mathcal{B}_{1+2}$ (13) attainable in product states for a two-body Hamiltonian from the studied family. For each N, violation of this bound with any separable state yields the presence of at least three-body terms in the Hamiltonian.

	N								
	2	3	4	5	6	7	8	9	10
$\mathcal{B}_{1+2}$	1.78	2.68	3.61	4.57	5.53	6.51	7.49	8.47	9.46

approach, in principle, can be performed for any k. Nevertheless, we will focus on the first physically interesting scenario presented above.

A. Example

As an example, consider an Ising chain on a complete interaction graph with a uniform external field in the z direction ($k \leqslant 2$). Tuning the field according to the coupling strength, as well as normalizing an entire Hamiltonian, we get

$$\mathcal{H}_I = \mathcal{N} J \left(\sum_{i<j}^{N} \sigma_z^i \sigma_z^j + \sum_{i}^{N} \sigma_z^i \right), \tag{15}$$

where $\mathcal{N} = 1/J(N+1)$. Suppose now that this system contains some amount of three-body interactions of the same symmetry and the actual Hamiltonian is, up to normalization, $\mathcal{H}^{(3)} = \mathcal{H}_I + \gamma_3 H_3$. In Fig. 3 we plot the maximal QFI in product states for different N and varying γ_3. Clearly, even for small γ_3, higher-order interaction is detected. Moreover, let us examine a case in which we set different values of $\gamma_3 = \{0.1, 0.5, 0.7, 1\}$ and, for each choice, compute the QFI in a random pure three-qubit product state. In this scenario, it is also possible to violate $\mathcal{B}_{1+2}$. Namely, for 10^5 samples the estimated frequency of violation is $\{0.23\%, 4.3\%, 6.1\%, 8\%\}$ for the respective choices of γ_3. Although, as expected, this frequency is small, it is still significant, and one can make conclusions about the present interaction type.

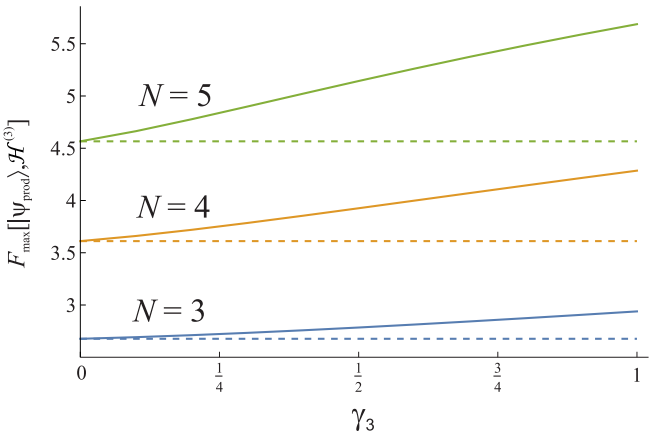

FIG. 3. Maximal quantum Fisher information in a product state for Hamiltonian $\mathcal{H}_I$ (15) with three-body contribution $\gamma_3 H_3$ for $N = 3, 4, 5$ and varying γ_3 (solid line). Bounds $\mathcal{B}_{1+2}$, which allow one to verify the presence of higher-order interactions, are plotted by dashed lines. One can see that it is possible to make statements about the order of interactions based on the presented results. If the bound of $\mathcal{B}_{1+2}$ is violated, then within the studied family of Hamiltonians, the interaction is at least 3-local.

An interesting thing to comment on is the change in eigenlevels structure. In general, the state that maximizes variance is given as $1/\sqrt{2}(|E_{\min}\rangle + |E_{\max}\rangle)$. This, however, does not need to be a product state, and indeed, it is not in most cases. Nevertheless, if the order of interaction increases, the structure of eigenlevels changes, and Heisenberg scaling is available with product states. In fact, increasing γ_3 causes an attraction of the lowest energy levels, resulting in stronger degeneracy when all couplings are equal. The resulting system is effectively a two-level structure, and it is possible to form many product states which are a uniform superposition of two levels; hence, the maximal value of variance and QFI can be achieved.

The discussed protocol could be especially useful for verifying whether the many-body couplings have been engineered in a quantum simulator or another setup without direct comparison of evolution with a specific k-local Hamiltonian. The importance of such tasks was discussed in the Introduction.

VI. POSSIBLE EXTENSIONS

Here, we will briefly discuss the problem of k-body-interaction verification outside of the discussed class of Hamiltonians. Let us consider the long-range-interaction transverse-field XY model

$$\mathcal{H}_{XY} = \mathcal{N} \left\{ J \sum_{i<j} \left[(1+\delta)\sigma_x^i \sigma_x^j + (1-\delta)\sigma_y^i \sigma_y^j \right] + b \sum_i \sigma_z^i \right\},$$

where $\mathcal{N}$ is a normalization constant, J is an exchange constant, δ is an anisotropy parameter, and b stands for an external magnetic field. For a similar discussion on the transverse-field Ising model on a complete interaction graph see [92]. The above Hamiltonian differs significantly from the ones studied before, and in principle, the conclusions drawn in the previous sections may not hold.

While working with the XY model we need to specify the free parameters. We choose to set $J = 1$ and scan over different δ and b. The choice of J is arbitrary due to the normalization and scanning over different field and anisotropy values. The physical range of δ is $[-1, 1]$. For large values of the external transverse field, the interaction terms' contribution decreases, and the results should converge to the case of local Hamiltonians. Thus, we choose to restrict b to a significant region of $\pm\sqrt{J^2 + \delta_{\max}^2}$, i.e., $b \in [-2, 2]$. Performing a numerical optimization over the set of pure product states for the case of $N = 3$, we obtain the data plotted in Fig. 4(a), where the step of the parameter change is chosen to be $1/5$. Note that for $N = 3$ particles, the long-range-interaction term is equivalent to the periodic boundary condition and makes it more feasible experimentally. In general, we observe that the maximal QFI in a product state, i.e., $\mathcal{B}_{XY}$, is given as 5.97. This corresponds to $\delta = 0.6$ and $b = \pm 1.6$. For the $\delta = 0$ cut (the XX model) the maximal

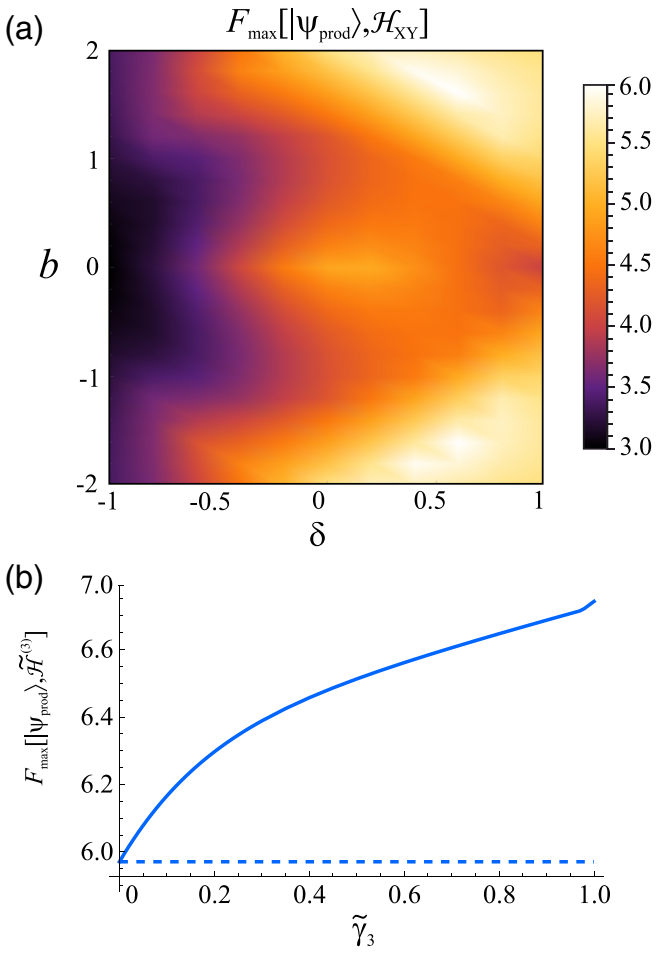

FIG. 4. (a) Maximal quantum Fisher information in a three-qubit product state for the normalized transverse-field XY Hamiltonian and different values of the external field b and anisotropy parameter δ. The free parameters were scanned within the region of ± 1 for δ and ± 2 for b with a step of $1/5$. The maximal QFI in a product state was found to be $\mathcal{B}_{XY} = 5.97$. For $\delta = 0$, the XX model, it yields a maximal value of 5.51. Since the QFI values plotted here do not saturate the upper bound of $N^2 = 9$, our reasoning can be used for higher-order-interaction verification. (b) Maximal quantum Fisher information in a product state for the normalized Hamiltonian $\tilde{\mathcal{H}}^{(3)} = \mathcal{H}_{XY} + \tilde{\gamma}_3 \tilde{H}_3$ with a varying $\tilde{\gamma}_3$ and the optimal XY model parameters $\delta = 0.6$ and $b = -1.6$ (solid line). The value of $\mathcal{B}_{XY}$ is plotted by a dashed line. Its violation allows one to verify that a three-body-interaction term is present in a system where one- and two-body terms are described with the XY model. Here, this is clearly possible.

QFI in a product state is found to be 5.51. Both of these numbers are essentially smaller than $N^2 = 9$, which is needed to verify the higher-order interactions. Later in this section we will consider $\mathcal{H}_{XY}$ with the optimal parameters $\delta = 0.6$ and $b = -1.6$.

First, we will examine a specific example of the many-body-interaction verification problem within the studied model. Then, a more general approach will be presented. Consider a three-body Hamiltonian $\tilde{H}_3$ which yields a QFI in a product state for the normalized $\tilde{\mathcal{H}}^{(3)} = \mathcal{H}_{XY} + \tilde{H}_3$ that violates $\mathcal{B}_{XY} = 5.97$. One possible choice is simply $\tilde{H}_3 = \sigma_x^1 \sigma_x^2 \sigma_x^3$, as it leads to the maximal QFI in a product state for $\tilde{\mathcal{H}}^{(3)}$ equal to 6.74. Similar to what we did in the previous section, we vary the amount of the $\tilde{H}_3$ contribution as a function of the coupling strength $\tilde{\gamma}_3$. Once again, we conclude that the presence of higher-order interaction terms can be verified through QFI, as illustrated in Fig. 4(b).

Choosing the three-body Hamiltonian in the XY model is far more arbitrary. Hence, an additional interesting question arises. What if the underlying many-body-interaction Hamiltonian could contain any three-body terms? For this reason, let us consider a three-body Hamiltonian of the general form $\tilde{H}_3 = \sum_{i,j,k=1}^{3} \gamma_{ijk} \sigma_i \otimes \sigma_j \otimes \sigma_k$. Now, we introduce small random couplings by sampling γ_{ijk} from a uniform distribution on $(-\gamma_{\max}, \gamma_{\max})$ and choose $\gamma_{\max} = 0.5$. This leads to multiple violations of $\mathcal{B}_{XY}$ for $\mathcal{N}(\mathcal{H}_{XY} + \tilde{H}_3)$ in the fixed optimal XY-model product state with a frequency of 1.4%, estimated on 10^5 runs. Consequently, it shows the ability of a more general three-body-interaction verification within the XY model using the QFI.

VII. CONCLUSIONS

We examined the possibility of detecting higher-order interactions with the use of quantum Fisher information. For normalized symmetric Ising-like Hamiltonians, we showed that the maximal QFI in product states is ordered with respect to the fixed interaction order. Moreover, we calculated the maximal QFI obtained in the product states for the most natural scenario in which one- and two-body terms are present. This allowed us to verify the presence of at least three-body interactions in the chosen family of Hamiltonians through the violation of this bound. As a possible extension, we analyzed an example concerning the three-body-interaction verification in the XY model.

The considered problem has a strong foundational meaning because it paves the way for a better understanding of the nature of interactions. Furthermore, it encompasses some practical applications for Hamiltonian learning and could provide new perspectives for many-body-interaction engineering. As argued before, we emphasize that QFI can be measured experimentally via multiple techniques.

Future research on this problem could focus on generalizing this observation to an arbitrary Hamiltonian class. We note that a different approach that contains no assumptions on the symmetry and interaction strength is possible. However, it would require the possibility of the mean energy measurement in an arbitrary state and a presumably unknown Hamiltonian.

Note added. Very recently, we became aware that a similar problem is being examined independently in a different manner by Bluhm *et al.* (see [96] for their proposed solution to this task).

ACKNOWLEDGMENTS

This research was supported by the National Science Centre (NCN, Poland) within Preludium Bis Project No. 2021/43/O/ST2/02679 (P.C. and W.L.) and OPUS Project No. 2023/49/B/ST2/03744 (T.S.).

APPENDIX: MAXIMAL QFI IN PRODUCT STATES

To perform an exact maximization of QFI for $N = 3$ and $k = 2$ let us consider the extremum conditions (9) explicitly:

$$G_{2,3}[1 + 2p_2(p_3 - 1) - 2p_3 + 2p_1(p_2 + p_3 - 1)] = 0,$$
$$G_{1,3}[1 + 2p_2(p_3 - 1) - 2p_3 + 2p_1(p_2 + p_3 - 1)] = 0,$$
$$G_{1,2}[1 + 2p_2(p_3 - 1) - 2p_3 + 2p_1(p_2 + p_3 - 1)] = 0,$$

where $G_{i,j} = 4(1 - p_i - p_j)$. Discarding the minimum solutions generated by eigenstates and limiting ourselves to $p_i \in [0, 1]$, from $G_{i,j}$ we get

$$p_1 = p_2 = p_3 = \tfrac{1}{2}, \tag{A1}$$

which is a local minimum with QFI equal to 3. Assuming that the first term is nonzero, we obtain

$$p_1 = \frac{1 + 2p_2(1 - p_3) + 2p_3}{2(1 - p_2 - p_3)}. \tag{A2}$$

This leads to QFI of 4 and can also be satisfied if all p_i are taken to be equal. Indeed, taking $p_i = p$ for all i, we get

$$F[|\psi\rangle^{\otimes 3}, H_2] = 2(-18p^4 + 36p^3 - 24p^2 + 3p), \tag{A3}$$

with a maximum of 4 for $p = 1/6(3 + \sqrt{3})$. For the three-body-interaction Hamiltonian H_3 the calculations can be performed in an alternative manner. The spectrum of H_3 consists of only two levels. The energy of $-1/3$ is associated with the eigenstates $|111\rangle, |100\rangle, |010\rangle, |001\rangle$ and $|000\rangle, |110\rangle, |101\rangle, |011\rangle$ for the corresponding eigenvalue of $1/3$. The maximal algebraically allowed variance is given as the square of the energy bandwidth $(E_{\max} - E_{\min})^2$. Most often, it is associated with highly entangled states; for the studied family of Hamiltonians it is a Greenberger-Horne-Zeilinger state. Here, however, due to additional degeneracies arising from the three-body couplings (see the discussion at the end of Sec. V), a product state that saturates the variance can be constructed. Taking a uniform superposition of $|000\rangle$ and $|001\rangle$, we obtain $|0\rangle \otimes |0\rangle \otimes 1/\sqrt{2}\,(|0\rangle + |1\rangle)$. From the symmetry of the Hamiltonian any permutation of $1/\sqrt{2}\,(|0\rangle + |1\rangle)$ among $|0\rangle$ is an equally valid solution.

Now, we give another example for $N = 4$ and $k = 2$. Here, we want to solve the set of four equations arising from (9). We do not report their explicit forms here, but one can easily generate them by calculating the variance in a parameterized product state. We find the following families of solutions to the given problem:

$$p_{i_1} = \frac{1}{2} - \frac{1}{\sqrt{2}}, \quad p_{i_2} = \frac{1}{2} - \frac{1}{\sqrt{2}}, \quad p_{i_3} = \frac{1}{2}, \quad p_{i_4} = \frac{1}{2}, \tag{A4a}$$

$$p_{i_1} = \frac{1}{2} + \frac{1}{\sqrt{2}}, \quad p_{i_2} = \frac{1}{2} + \frac{1}{\sqrt{2}}, \quad p_{i_3} = \frac{1}{2}, \quad p_{i_4} = \frac{1}{2}, \tag{A4b}$$

$$p_{i_1} = p_{i_2} = \frac{1}{2} - \frac{1}{\sqrt{2}}, \quad p_{i_3} = p_{i_4} = \frac{1}{2} + \frac{1}{\sqrt{2}}, \tag{A4c}$$

$$p_1 = p_2 = p_3 = p_4 = \frac{1}{2}, \tag{A4d}$$

$$p_{i_1} = \frac{1}{2} - \frac{1}{\sqrt{10}}, \quad p_{i_2} = \frac{1}{2} - \sqrt{\frac{5}{2}}, \quad p_{i_3} = \frac{1}{2}, \quad p_{i_4} = \frac{1}{2}, \tag{A4e}$$

$$p_{i_1} = \frac{1}{2} - \frac{1}{\sqrt{10}}, \quad p_{i_2} = \frac{1}{2} - \sqrt{\frac{5}{2}}, \quad p_{i_3} = \frac{1}{2}, \quad p_{i_4} = \frac{1}{2} \tag{A4f}$$

$$p_{i_1} = \frac{1}{2} + \frac{1}{\sqrt{10}}, \quad p_{i_2} = \frac{1}{2} + \sqrt{\frac{5}{2}}, \quad p_{i_3} = \frac{1}{2}, \quad p_{i_4} = \frac{1}{2}, \tag{A4g}$$

$$p_1 = p_2 = p_3 = p_4 = \frac{1}{2} \pm \frac{1}{\sqrt{10}}, \tag{A4h}$$

where the trivial (eigenstate) solutions were discarded. These families are indexed with (i_1, i_2, i_3, i_4), which take distinct values from [1,4]. This resembles the symmetric structure of the studied Hamiltonians. One can check directly that the last solution with $\forall_i\, p_i = p$ gives rise to the highest QFI, as reported in the main text, i.e., $F_{\max}[|\psi_{\text{prod}}\rangle, H_2] = 24/5 = 4.8$.

For $N > 3$, solutions of the form (A2) do not guarantee the vanishing derivatives for all parameters, as shown for $N = 4$. Nevertheless, the more complicated solutions also contain the one for which the optimal state is a tensor product of single-qubit states (see the main text). To further check our results and examine $N \geqslant 5$, we found the upper bound on the QFI with standard numerical and symbolical optimization techniques built in Wolfram *Mathematica* (version 13.3). All of the results obtained by solving the extremum conditions are in agreement with the numerical calculations. It is worth noting that the problem under consideration is exactly solvable. One can express the variance of a Hamiltonian on two state copies, i.e., $\mathrm{Tr}[(\mathbb{1} \otimes H^2 - H \otimes H)\rho']$, with $\rho' = \rho \otimes \rho$, which reduces it to a linear form. Constraints on the reduced states can ensure that the maximization is performed over the separable states ρ and hence solves the problem.

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