Solutions of the Logarithmic Schrödinger Equation in a Rotating Harmonic Trap

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We study the influence of the nonlinearity in the Schrödinger equation on the motion of quantum particles in a harmonic trap. In order to obtain exact analytic solutions, we have chosen the logarithmic nonlinearity. The unexpected result of our study is the existence in the presence of nonlinearity of two or even three coexisting Gaussian solutions.

Keywords: nonlinear Schrödinger equation, rotating harmonic trap, logarithmic Schrödinger equation, exact solutions of a nonlinear Schrödinger equation

I. INTRODUCTION

The nonlinear Schrödinger equation with the logarithmic nonlinearity (we use the units $\hbar = 1$ and $m = 1$)

$$i\partial_t\psi(\mathbf{r},t) = \left(-\frac{1}{2}\Delta + V(\mathbf{r},t) - b\log(|\psi(\mathbf{r},t)|^2/a^3)\right)\psi(\mathbf{r},t) \tag{1}$$

was introduced [Bialynicki-Birula and Mycielski] long time ago to seek possible departures of quantum mechanics from the linear regime. The parameter b measures the strength of the nonlinear interaction (positive b means attraction) and a is needed to make the argument of the logarithm dimensionless — it plays no significant role since the change of a results only in an additive constant to the potential. In what follows, we shall absorb the parameter a into the wave function that amounts effectively to putting $a = 1$.

It has been proven in beautiful experiments with neutron beams [Shimony, Shull et al., Gähler et al.] that the nonlinear effects in quantum mechanics, if they exist at all, are extremely small. The upper limit for the constant b was determined to be 3.3 10^{-15} eV. Thus, the applicability of the logarithmic Schrödinger equation to the time evolution of wave functions seems to have been ruled out. Nevertheless this equation, owing to its unique mathematical properties, has been used in many branches of physics to model the nonlinear behavior of various phenomena. It has been applied in the study of dissipative systems [Hernandez and Remaud], in nuclear physics [Hefter], in optics [Krolikowski et al., Buljan et al.], and even in geophysics [De Martino et al.]. In contrast to the properties of other nonlinear equations, the logarithmic Schrödinger equation in any number of dimensions possesses analytic solutions, called Gaussons in [Bialynicki-Birula and Mycielski]. Gaussons represent localized nonspreading solutions of the Gaussian shape. The internal structure of the Gaussons may also change in time. The existence of these analytic solutions enables one to study in detail the influence of nonlinearities. In this paper we focus our attention on the behavior of the solutions of the logarithmic Schrödinger equation in a rotating harmonic trap. The aim of our study was to see to what extent the nonlinear interaction may change the dynamics and affect the stability of solutions. Perhaps, our results will help to better understand the behavior of the Bose-Einstein condensate in a rotating trap. Previous studies of these problems (for example, [Recati et al.] and [Cozzini et al.]) were often based on the hydrodynamic equations and we plan in the future to express our results in terms of the hydrodynamic variables.

II. FORMULATION OF THE PROBLEM

The logarithmic Schrödinger equation in a rotating trap has the form

$$i\partial_t\psi(\mathbf{r},t) = \left(-\frac{1}{2}\Delta + \frac{1}{2}\mathbf{r}\cdot\hat{V}(t)\cdot\mathbf{r} - b\log(|\psi(\mathbf{r},t)|^2)\right)\psi(\mathbf{r},t), \tag{2}$$

where the symmetric 3×3 matrix $\hat{V}(t)$ depends on time due to rotation. In order to simplify the analysis of stability, we assume that the trap is subjected to a uniform rotation and we shall use the coordinate system co-rotating with the trap. In this manner the potential becomes time-independent but due to rotation there appears an additional term in the equation.

$$i\partial_t\psi(\mathbf{r},t) = \left(-\frac{1}{2}\Delta + \frac{1}{2}\mathbf{r}\cdot\hat{V}\cdot\mathbf{r} - b\log(|\psi(\mathbf{r},t)|^2) - \mathbf{\Omega}\cdot\mathbf{M}\right)\psi(\mathbf{r},t), \tag{3}$$

where $\mathbf{\Omega}$ is the vector of angular velocity and $\mathbf{M} = \mathbf{r}\times\mathbf{p}$ is the operator of angular momentum. We shall seek the solutions of Eq. (3) in the Gaussian form

$$\psi(\mathbf{r},t) = N(t)e^{if(t)}\exp\left(-\frac{1}{2}\tilde{\mathbf{r}}\cdot(\hat{A}(t) + i\hat{B}(t))\cdot\tilde{\mathbf{r}}(t) + i\boldsymbol{\pi}(t)\cdot\mathbf{r}\right), \tag{4}$$

where $\tilde{\mathbf{r}} = \mathbf{r} - \boldsymbol{\xi}(t)$. The time-dependent vectors $\boldsymbol{\xi}(t)$ and $\boldsymbol{\pi}(t)$ specify the position and momentum of the center of mass of the Gaussian wave packet and the time-dependent real symmetric matrices $\hat{A}(t)$ and $\hat{B}(t)$ specify the shape and the internal motion of the wave packet, respectively. The two real functions $N(t)$ and $f(t)$ define the normalization and the overall phase of the Gausson. Substituting this Ansatz into Eq. (3), we arrive at the following set of *ordinary* differential equations for all the functions entering our formula (4)

$$\frac{d\hat{A}(t)}{dt} = \hat{B}(t)\hat{A}(t) + \hat{A}(t)\hat{B}(t) - \left[\hat{\Omega}, \hat{A}(t)\right], \tag{5}$$

$$\frac{d\hat{B}(t)}{dt} = \hat{B}(t)^2 - \hat{A}(t)^2 + \hat{V} + 2b\hat{A}(t) - \left[\hat{\Omega}, \hat{B}(t)\right], \tag{6}$$

$$\frac{d\boldsymbol{\xi}(t)}{dt} = \boldsymbol{\pi}(t) - \boldsymbol{\Omega} \times \boldsymbol{\xi}(t), \tag{7}$$

$$\frac{d\boldsymbol{\pi}(t)}{dt} = -\hat{V}\cdot\boldsymbol{\xi}(t) - \boldsymbol{\Omega} \times \boldsymbol{\pi}(t), \tag{8}$$

$$\frac{dN(t)}{dt} = \frac{1}{2}\mathrm{Tr}\{\hat{B}(t)\}N(t), \tag{9}$$

$$\frac{df(t)}{dt} = -\frac{1}{2}\left(\mathrm{Tr}\{\hat{A}(t)\} + \boldsymbol{\pi}(t)\cdot\boldsymbol{\pi}(t) - \boldsymbol{\xi}(t)\cdot\hat{V}\cdot\boldsymbol{\xi}(t)\right), \tag{10}$$

where the antisymmetric matrix $\hat{\Omega}$ and the components of the angular velocity vector $\boldsymbol{\Omega}$ are related through the formula $\Omega_{ij} = \epsilon_{ijk}\Omega^k$. Note, that the internal motion (described by $\hat{A}(t)$ and $\hat{B}(t)$) completely decouples from the motion of the center of mass (described by $\boldsymbol{\xi}(t)$ and $\boldsymbol{\pi}(t)$). In turn, the equations for the normalization factor and the phase can be integrated after the internal and the center of mass motion has been determined. This decoupling follows from the general theorem [García-Ripoll et al.] and [Bialynicki-Birula and Bialynicka-Birula] stating that from every solution of a nonlinear Schrödinger equation in a harmonic potential (including time-dependent potential) one may obtain a solution displaced by a classical trajectory fully preserving the shape of the wave function.

III. SOLUTIONS AND THEIR STABILITY

In what follows, for simplicity, we shall assume that the trap rotates along one of its principal axis. In this case the motion in the direction perpendicular to the rotation plane decouples and we are left with a two-dimensional problem. In the stationary state of our system the center of mass motion must be absent ($\boldsymbol{\xi}(t) = 0, \boldsymbol{\pi}(t) = 0$) The stationary state of the system is described by the wave function characterized by the solution of the following two time-independent equations for two 2×2 matrices A and B

$$0 = \hat{B}\hat{A} + \hat{A}\hat{B} - \left[\hat{\Omega}, \hat{A}\right], \tag{11}$$

$$0 = \hat{B}^2 - \hat{A}^2 + \hat{V} + 2b\hat{A} - \left[\hat{\Omega}, \hat{B}\right]. \tag{12}$$

We shall seek the solutions of these equations in the coordinate frame in which the matrix $\hat{V}$ is diagonal, $\hat{V} = \mathrm{Diag}\{\omega_1^2, \omega_2^2\}$. We assume, for definitness, that $\omega_1 < \omega_2$. It follows from Eqs. (11–12) that in this frame the matrix $\hat{A}$ is also diagonal and the matrix $\hat{B}$ is off-diagonal. Finally, we are left with three equations for two matrix elements α_1, α_2 of $\hat{A}$ and one matrix element β of $\hat{B}$

$$(\alpha_1 + \alpha_2)\beta - (\alpha_1 - \alpha_2)\Omega = 0, \tag{13}$$

$$\beta^2 - \alpha_1^2 + \omega_1^2 + 2b\alpha_1 + 2\beta\Omega = 0, \tag{14}$$

$$\beta^2 - \alpha_2^2 + \omega_2^2 + 2b\alpha_2 - 2\beta\Omega = 0. \tag{15}$$

It follows from Eq. (13) that in the absence of rotation β must vanish and we obtain immediately two physically acceptable solutions of the decoupled quadratic equations for the parameters α

$$\alpha_1 = (\omega_1\sqrt{1 + b^2/\omega_1^2} + b), \tag{16}$$

$$\alpha_2 = (\omega_2\sqrt{1 + b^2/\omega_2^2} + b), \tag{17}$$

$$\beta = 0. \tag{18}$$

The two remaining solutions yield negative values of the α's and must be rejected. Thus, in the absence of rotation the nonlinearity modifies only the size of the Gaussian wave function without introducing any significant changes. Even for negative values of b (nonlinear repulsion), stable solutions described by (16) and (17) always exist, no matter how strong is the repulsion.

Simple analytic formulas can also be obtained in the presence of rotation but without nonlinearity. The formulas for the Gausson parameters read in this case

$$\alpha_1 = \frac{\sqrt{\omega_1^2+\omega_2^2+2\Omega^2 \pm 2\sqrt{(\omega_1^2-\Omega^2)(\omega_2^2-\Omega^2)}}}{1+\sqrt{(\omega_2^2-\Omega^2)/(\omega_1^2-\Omega^2)}}, \tag{19}$$

$$\alpha_2 = \frac{\sqrt{\omega_1^2+\omega_2^2+2\Omega^2 \pm 2\sqrt{(\omega_1^2-\Omega^2)(\omega_2^2-\Omega^2)}}}{1+\sqrt{(\omega_1^2-\Omega^2)/(\omega_2^2-\Omega^2)}}, \tag{20}$$

$$\beta = \Omega\frac{1-\sqrt{(\omega_2^2-\Omega^2)/(\omega_1^2-\Omega^2)}}{1+\sqrt{(\omega_2^2-\Omega^2)/(\omega_1^2-\Omega^2)}}. \tag{21}$$

The values of α are real in the two regions of stability when $\Omega < \omega_1$ (region 1) and $\Omega > \omega_2$ (region 2). The same regions of stability were obtained in the analysis of the characteristic frequencies in classical or quantum-mechanical center-of-mass motion [Bialynicki-Birula and Bialynicka-Birula]. In the formulas (19) and (20) the + and – sign is to be chosen for the region 1 and the region 2, respectively.

In the presence of both rotation and nonlinearity the properties of solutions change significantly. The most striking difference is the appearance of additional stationary Gaussian solutions. This is an unexpected result because in the linear theory a purely Gaussian shape always is found for *only one* fundamental state of the system — all other states have polynomial prefactors. We have not been able to find closed expressions for the parameters α and β, so we had to resort to numerical analysis of the solutions of Eqs. (13–15). We present our results in three plots showing the calculated values of the parameters α_1 and α_2 that determine the shape of the Gaussian wave function. These values are plotted as functions of the angular velocity Ω. In all plots we have fixed the trap parameters to be $\omega_1 = \sqrt{2/3}, \omega_2 = \sqrt{4/3}$. We have chosen three values of b to describe the following characteristic cases. In Fig. 1 we plot the values of α's without the nonlinear interaction ($b = 0$). In Fig. 2 we added the attractive nonlinear interaction ($b = 1$) and in Fig. 3 the repulsive nonlinear interaction ($b = -1$).

IV. CONCLUSIONS

Knowing the exact analytic form of the solutions of our nonlinear Schrödinger equation we were able determine the influence of rotation and nonlinearity on the stability of solutions. The unexpected result of our analysis is that the repulsive interaction *expands* the region of stability. We have to admit, however, that this may be true only for the special form of the nonlinearity: the logarithmic nonlinearity.

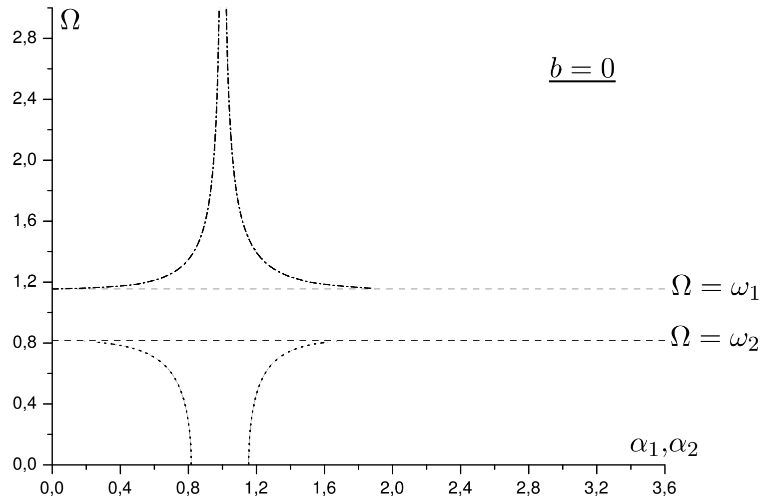

FIG. 1: This plot shows the values of α_1 and α_2 in the absence of the nonlinearity. For each value of Ω in the stability regions there is just one Gaussian wave function whose shape is described by the values of α's.

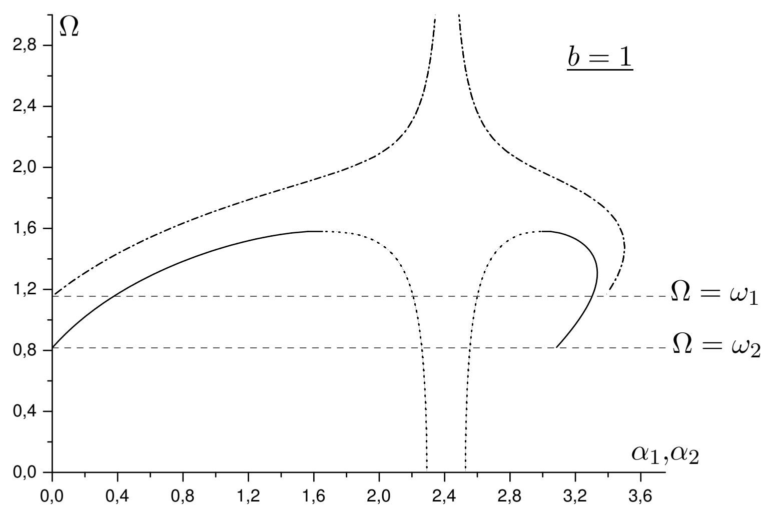

FIG. 2: This plot shows the values of α_1 and α_2 in the case of attractive nonlinear interaction. For sufficiently large values of b, as in this case, there are no regions of instability. For small and for large values of Ω there is just one Gaussian wave function but for intermediate values there are two or even three solutions. Matching pairs of α's are distinguished by the lines of the same style: solid, dashed, and dotted.

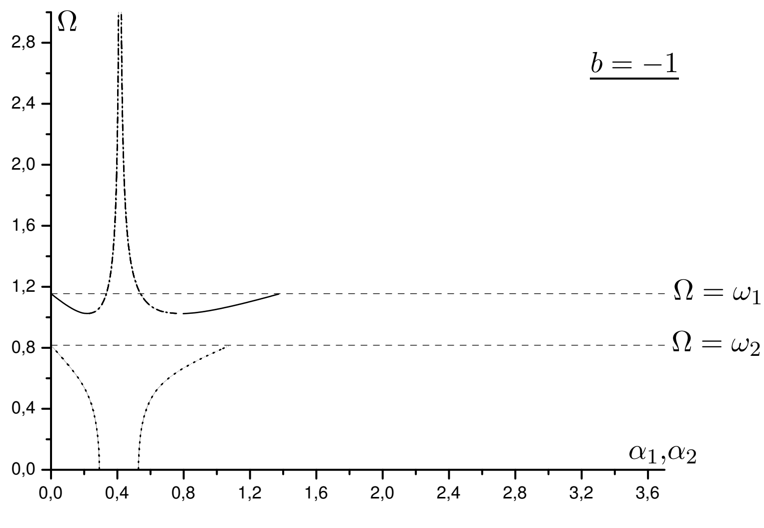

FIG. 3: This plot shows the values of α_1 and α_2 in the case of repulsive nonlinear interaction. The upper region of stability is extended now downwards as compared to the case without the nonlinear term. Moreover, there are two solutions (solid and dashed lines) that coexist in the newly established region of stability.

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