# Classical-quantum correspondence in a rotating harmonic trap

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**Abstract**

Complete description of the classical and quantum dynamics of a particle in an anisotropic, rotating, harmonic trap is given. The analysis of the quantum-mechanical problem is made simple due to a direct connection between the classical mode vectors and the quantum-mechanical wave functions. This connection is obtained via the matrix Riccati equation that governs the time evolution of squeezed states of the harmonic oscillator. This observation enabled us to give a **direct prescription how to construct a complete set of wave functions of the quantum stationary states of our physical model only from the classical trajectories**.

## Introduction

Connection between classical and quantum description of physical system manifests itself in a very amazing and nontrivial way. We understand how to describe any system in classical and quantum language and also we believe that quantum description should smoothly transform to classical one when we neglect quantum corrections. But in general we still do not understand how one can make this transformation in practice and find the classical motion in the quantum one.

The question which we answer in this paper is how this exact connection between classical and quantum dynamics for the rotating, anisotropic, harmonic trap is realized in the Schrödinger picture where the whole dynamics is contained in the evolution of the wave function of the system.

## Classical dynamics

### Hamiltonian

First we perform the transformation to the rotating frame. In the rotating frame the Hamiltonian has the form

$$\boxed{\mathcal{H} = \frac{\vec{p}^2}{2m} + \vec{r}\cdot\hat{\Omega}\cdot\vec{p} + \frac{m}{2}\vec{r}\cdot\hat{V}\cdot\vec{r}} \tag{1}$$

The potential matrix $\hat{V}$ is symmetric and positive definite. The angular velocity matrix $\hat{\Omega}$ is related to the components of the angular velocity vector through the formula $\Omega_{ik} = \epsilon_{ijk}\Omega^j$. Equations of motion have a form

$$\frac{d\vec{r}}{dt} = \frac{\vec{p}}{m} - \hat{\Omega}\cdot\vec{r}, \tag{2a}$$
$$\frac{d\vec{p}}{dt} = -m\hat{V}\cdot\vec{r} - \hat{\Omega}\cdot\vec{p}. \tag{2b}$$

### General physical solution

Because the equations (2) are linear, any physical solution can be expressed in the following way

$$\begin{pmatrix}\vec{r}(t)\\ \vec{p}(t)\end{pmatrix} = \sum_{i=1}^{3}\left[\lambda_i\begin{pmatrix}\vec{R}_i\\ \vec{P}_i\end{pmatrix}e^{i\omega_i t} + \lambda_i^*\begin{pmatrix}\vec{R}_i^*\\ \vec{P}_i^*\end{pmatrix}e^{-i\omega_i t}\right], \tag{3}$$

where the coefficients $\lambda_i$ are determined by the initial conditions. The characteristic frequencies $\omega_k$ and the amplitudes $\left(\vec{R}_i, \vec{P}_i\right)$ obey the following matrix equation

$$\begin{pmatrix}-\hat{\Omega} - i\omega_i & \frac{1}{m}\\ -m\hat{V} & -\hat{\Omega} - i\omega_i\end{pmatrix}\cdot\begin{pmatrix}\vec{R}_i\\ \vec{P}_i\end{pmatrix} = 0. \tag{4}$$

## Quantum Dynamics

In the quantum case, the dynamics of the system is dictated by the Schrödinger equation

$$\boxed{i\hbar\partial_t\Psi(\vec{r},t) = \left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + \frac{\hbar}{i}\vec{r}\cdot\hat{\Omega}\cdot\vec{\nabla} + \frac{m}{2}\vec{r}\cdot\hat{V}\cdot\vec{r}\right)\Psi(\vec{r},t).}$$

### Gaussian wave function

In the first step let us consider the dynamics of a state described by a Gaussian wave function

$$\Psi(\vec{r},t) = N(t)e^{\frac{i}{\hbar}\phi(t)}e^{-\frac{m}{2\hbar}\left[\vec{r}-\vec{R}(t)\right]\cdot\hat{K}(t)\cdot\left[\vec{r}-\vec{R}(t)\right]+\frac{i}{\hbar}\vec{r}\cdot\vec{P}(t)}, \tag{5}$$

where the matrix $\hat{K}(t)$ is of course symmetric and its real part is positive.

Parameters $\vec{R}$ and $\vec{P}$ have a direct interpretation as the position and momentum of the center of the wave function.

### Evolution of parameters

From the Schrödinger equation follows that equations for the parameters of the wave function (5) have a form

$$\frac{d\hat{K}(t)}{dt} = -i\hat{K}(t)^2 + i\hat{V} - \left[\hat{\Omega},\hat{K}(t)\right], \tag{6a}$$
$$\frac{d\vec{R}(t)}{dt} = \frac{\vec{P}(t)}{m} - \hat{\Omega}\cdot\vec{R}(t), \tag{6b}$$
$$\frac{d\vec{P}(t)}{dt} = -m\hat{V}\cdot\vec{R}(t) - \hat{\Omega}\cdot\vec{P}(t). \tag{6c}$$

Comparing the equations (6b) and (6c) with the classical equations of motion (2), one can see that the dynamics of the center of the wave packet is the same as the dynamics of a classical particle. Obviously it is a realization of the Ehrenfest theorem which is exactly satisfied for linear systems.

### Evolution of the shape

The shape of our state is described by the equation (6a) and it has the form of a **matrix Riccati equation**. Following the standard procedure of solving such equations, we shall search for its solutions in the form

$$\hat{K}(t) = -\frac{i}{m}\hat{N}(t)\cdot\hat{D}^{-1}(t), \tag{7}$$

where the matrices $\hat{N}$ and $\hat{D}$ obey the following *linear equations*

$$\frac{d\hat{N}}{dt} = -m\hat{V}\cdot\hat{D} - \hat{\Omega}\cdot\hat{N}, \tag{8a}$$
$$\frac{d\hat{D}}{dt} = \frac{1}{m}\hat{N} - \hat{\Omega}\cdot\hat{D}. \tag{8b}$$

The linearization of the Riccati equation leads to a direct relationship between classical and quantum theory. Comparing Eqs. (8) with Eqs. (2), one can see that the columns of the matrices $\hat{N}$ and $\hat{D}$ satisfy the same equations as the classical position and momentum vectors, respectively. Therefore, from the knowledge of the classical motion, one may determine the evolution of Gaussian wave function. It is a desirable manifestation of an exact connection between classical and quantum mechanics in the language of wave functions.

### Stationary Gaussian state

To find a stationary Gaussian state

$$\boxed{\Psi_0(\vec{r}) \sim \exp\left[-\frac{m}{2\hbar}\vec{r}\cdot\hat{K}_0\cdot\vec{r}\right]} \tag{9}$$

we have to solve the following algebraic matrix Riccati equation

$$0 = -i\hat{K}_0^2 + i\hat{V} - \left[\hat{\Omega},\hat{K}_0\right], \tag{10}$$

where the matrix $\hat{K}_0$ describes the shape of a stationary Gaussian state.

It is worth to notice that if we find two matrices $\hat{D}(t)$ and $\hat{N}(t)$ which satisfy the equations (8) and have a following form:

$$\hat{D}(t) = \hat{D}_0\cdot\hat{E}(t), \tag{11}$$
$$\hat{N}(t) = \hat{N}_0\cdot\hat{E}(t). \tag{12}$$

then the matrix $\hat{K}_0$ defined by the equation

$$\hat{K}_0 = -\frac{i}{m}\hat{N}_0\cdot\hat{D}_0^{-1} \tag{13}$$

is a solution of the equation (10).

The problem of finding such matrices is not a hard task. If we take the classical eigenmodes as the columns of these matrices, then the matrix $\hat{E}(t)$ will simply be a diagonal matrix with the elements $e^{i\omega_i t}$ and the matrices $\hat{D}_0$ and $\hat{N}_0$ can be build from the amplitudes $\vec{R}_i$ and $\vec{P}_i$ of the classical modes.

Of course it is still not clear which three modes from the six classical one should use in this construction. We must remember that we want the matrix $\hat{K}_0$ to describe a real Gaussian and therefore one should use such a set of classical modes that create a matrix $\hat{K}_0$ with a positive real part. One can show that there is at most one such set.

### Other stationary states

It follows from the equations (6) that the motion of a Gaussian state with a constant shape $\hat{K}(t) = \hat{K}_0$, centered on the classical trajectory, is obviously possible. Such a situation is described by the wave function

$$\boxed{\Psi(\vec{r},t) = N(t)e^{\frac{i}{\hbar}\phi(t)}e^{-\frac{m}{2\hbar}\left[\vec{r}-\vec{R}(t)\right]\cdot\hat{K}_0\cdot\left[\vec{r}-\vec{R}(t)\right]+\frac{i}{\hbar}\vec{r}\cdot\vec{P}(t)},} \tag{14}$$

where the vectors $\vec{R}(t)$ and $\vec{P}(t)$ obey the classical equations of particle motion (2). Let us choose as a solution the physical trajectory generated by the two conjugated modes of the system (3)

$$\vec{R}(t) = \lambda\vec{R}_i e^{i\omega_i t} + \lambda^*\vec{R}_i^* e^{-i\omega_i t}, \tag{15a}$$
$$\vec{P}(t) = \lambda\vec{P}_i e^{i\omega_i t} + \lambda^*\vec{P}_i^* e^{-i\omega_i t}. \tag{15b}$$

The coefficient $\lambda$ is a scale factor of the classical trajectory. In this situation, the wave function has the following form

$$\Psi(\vec{r},t) = Ne^{-\chi}e^{-i\Omega_0 t}\exp\left[-\beta^2 e^{-2i\omega_i t} + 2\vec{\alpha}\cdot\vec{r}e^{-i\omega_i t}\right]\Psi_0(\vec{r}), \tag{16}$$

where

$$\Omega_0 = \frac{1}{2}\mathrm{Tr}(\Re\hat{K}_0), \tag{17a}$$
$$\chi = \frac{1}{4\hbar\omega_i m}\left(\vec{P}_i^2 - m^2\vec{R}_i\cdot\hat{V}\cdot\vec{R}_i\right), \tag{17b}$$
$$\vec{\alpha} = \frac{\lambda}{2\hbar}\left(m\hat{K}_0\cdot\vec{R}_i + i\vec{P}_i\right), \tag{17c}$$
$$\beta^2 = \frac{\lambda^2}{4\hbar\omega_i m}\left[m^2\vec{R}_i\cdot\left(2\omega_i\hat{K}_0 - \hat{V}\right)\cdot\vec{R}_i + \vec{P}_i^2\right]. \tag{17d}$$

When we use the formula for the generating function of the Hermitte polynomials $H_n(\xi)$

$$\boxed{e^{-z^2+2\xi z} = \sum_{n=0}^{\infty}H_n(\xi)\frac{z^n}{n!}} \tag{18}$$

one obtains a decomposition into stationary states. Indeed, if we take in (16) $\xi = \vec{\alpha}\cdot\vec{r}/\beta$ and $z = \beta e^{-i\omega_i t}$, the expansion (18) becomes

$$\Psi(\vec{r},t) = Ne^{-\chi}e^{-i\Omega_0 t}\sum_{n=0}^{\infty}\frac{\beta^n}{n!}H_n\left(\frac{\vec{\alpha}\cdot\vec{r}}{\beta}\right)e^{-in\omega_i t}\Psi_0(\vec{r}). \tag{19}$$

Our wave function is a superposition of the stationary states - the states whose evolution in time appears only in the evolution of the phase. For each $n$, we have defined exactly one stationary state - the $n$-th excitation of the $i$-th mode of the system

$$\Psi_n^{(i)}(\vec{r}) = H_n\left(\frac{\vec{\alpha}\cdot\vec{r}}{\beta}\right)\Psi_0(\vec{r}). \tag{20}$$

It is very easy to show that the eigenvalue of the Hamiltonian in this state is $E_n^{(i)} = \hbar n\omega_i + \hbar\Omega_0$.

To obtain a whole set of stationary states, one should repeat this construction for each mode, and because of the properties of the Hermitte polynomials completeness of this set is guaranteed.

## Conclusions

We have presented a full classical and quantum analysis of the dynamical properties of an anisotropic, rotating harmonic trap. **The main result of this analysis is the discovery of a direct link between the solutions of the Newton equations and the Schrödinger equation**. We gave a prescription how to construct a complete set of wave functions of the quantum stationary states from the classical trajectories.

## References

1. T. S. **(quant-ph/0608130)**
2. T. S. and I. Białynicki-Birula, **(quant-ph/0409070)**

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