Homework 1 and 2

Set number 1 - basics, single qubit

Problem 1

The spin 1/2 operators are given by $\hat{S}_k = \hat{\sigma}_k/2$, where σ_k operators have eigenvalues equal to ± 1. In the basis of eigenstates of $\hat{S}_z$ these operators are given by Pauli matrices:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$
$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ,$$
$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1}$$

Find the eigenstates of $\hat{\sigma}_x$ and $\hat{\sigma}_y$ (doing this for $\hat{\sigma}_z$ is trivial, I hope). Check the completeness relation of eigenstates of $\hat{\sigma}_x$ or $\hat{\sigma}_y$. Take one of eigenstates of $\hat{\sigma}_z$ and express it in basis of eigenstates of $\hat{\sigma}_x$ or $\hat{\sigma}_y$. What are the consequences of these results for the following experiment: a spin is measured (in an ideal way) to be in $+\frac{1}{2}$ state along the z axis, and immediately after such a measurement another one is made, sensing the projection of the spin on x or y axis. What are the probabilities of getting the two possible results?

Problem 2

We all like the basis $|\pm z\rangle$ of eigenstates of $\hat{\sigma}_z$ (usually we call these states $|+z\rangle = |0\rangle$ and $|-z\rangle = |1\rangle$). However, one can encounter a situation in which using a different basis is convenient. Let us use the basis of eigenstates of $\hat{\sigma}_x$ consisting of $|+x\rangle$ and $|-x\rangle$. What is the matrix representation of the old $\hat{\sigma}_z$ operator in this basis? Use the definition of matrix elements of a given operator in a given basis to obtain the result. Then obtain the unitary matrix $\hat{U}$ which transforms between the two bases:

$$|\pm x\rangle = \hat{U}\,|\pm z\rangle \ , \tag{2}$$

and calculate the matrix given by

$$\hat{U}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{U} \ , \tag{3}$$

which should be the $\hat{\sigma}_z$ operator transformed from $|\pm z\rangle$ basis to $|\pm x\rangle$ basis.

Problem 3

Let us take the pure state of a two-level system:

$$|\Psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle \ . \tag{4}$$

Remembering that $\hat{\sigma}_z\,|0\rangle = |0\rangle$ and $\hat{\sigma}_z\,|1\rangle = -\,|1\rangle$ calculate the expectation values of Pauli operators:

$$\langle\Psi|\,\hat{\sigma}_k\,|\Psi\rangle \ \text{ where } k = x, y, z \ . \tag{5}$$

These are of course the three components of the Bloch vector.

Problem 4

Show that state

$$|\Psi_+\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \ , \tag{6}$$

is an eigenstate corresponding to $+\Omega/2$ eigenvalue of

$$\hat{H} = \frac{\Omega}{2}\vec{\sigma}\cdot\vec{n} \tag{7}$$

where $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. What is the $|\Psi_-\rangle$ state that corresponds to $-\Omega/2$ eigenvalue?

Problem 5 - some fun with rotation operator

The operator

$$e^{-i\vec{n}\cdot\vec{\sigma}\phi/2} = \cos(\phi/2) - i\vec{n}\cdot\vec{\sigma}\sin(\phi/2) \ , \tag{8}$$

"rotates" the state of a two-level system by angle ϕ about an axis given by unit length vector $\vec{n}$. This means that if $\vec{v}_\Psi$ is the Bloch vector corresponding to state $|\Psi\rangle$, then the Bloch vector $\vec{v}_{\Psi'}$ of the state $|\Psi'\rangle = e^{-i\vec{n}\cdot\vec{\sigma}\phi/2}|\Psi\rangle$ is obtained from $\vec{v}_\Psi$ by such a rotation. Make sure that it really works in this way by calculating a few examples. Rotate the $|+z\rangle$ state by $\pi/2$ and π about the x axis. Do the same with the general $|\Psi\rangle$ state. Try a rotation by a general angle ϕ about x or y axis. You should convince yourself that the final and initial Bloch vectors are connected by an appropriate 3×3 rotation matrix. You can find the rotation matrices for real 3D vectors at *https://en.wikipedia.org/wiki/Rotation_matrix*. What happens when a quantum state gets rotated by 2π about some axis? Can this result have measurable consequences?

Set number 2 - two qubits, entanglement, reduced density matrix, measurements on bipartite states

Problem 2.1

Write the matrices of $\hat{\sigma}_x\otimes\hat{\sigma}_x$, $\hat{\sigma}_y\otimes\hat{\sigma}_y$, and $\hat{\sigma}_z\otimes\hat{\sigma}_z$ operators in the basis of $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$ states (the first operator in the tensor product corresponds to qubit A, while the second corresponds to qubit B, the same holds for indices in the state vectors, $|i_A j_B\rangle$).

Problem 2.2

Check that the state

$$|\chi_+\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \ , \tag{9}$$

is separable. This means that it can be written as

$$|\chi_+\rangle = |\phi\rangle_A \otimes |\phi\rangle_B \ , \tag{10}$$

where $|\phi\rangle_A$ and $|\phi\rangle_B$ are states of qubits A and B. What are these states?

Now look at the state in which one of the amplitudes has a negative sign:

$$|\chi_-\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) \ . \tag{11}$$

Check that this state is not separable, i.e. it is entangled. One criterion was discussed in the first lecture on entanglement, the other will be discussed this week.

Problem 2.3

Assume that the Hamiltonian of the two qubits is given by

$$\hat{H} = \frac{K_{AB}}{4}(\mathbb{I}_A - \hat{\sigma}_{z,A}) \otimes (\mathbb{I}_B - \hat{\sigma}_{z,B}) \ , \tag{12}$$

where $\mathbb{I}_{A,B}$ is the unity operator in the given space. Write down the matrix of this Hamiltonian. Can you see how the evolution due to such an interaction can turn the $|\chi_+\rangle$ state into $|\chi_-\rangle$ state? (these are the states from the previous problem)

Problem 2.4

Take the state of the whole system to be

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B \pm |\downarrow\rangle_A \otimes |\uparrow\rangle_B) \ , \tag{13}$$

Calculate the reduced density matrix of spin A. Repeat for spin B. I guess that repeating all this for $|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\uparrow\rangle_B \pm |\downarrow\rangle_A \otimes |\downarrow\rangle_B)$ would be boring, but make sure that you can see how this will work for them.

Problem 2.5

While working in the standard basis of spin up and down along the z axis, write down a density matrix corresponding to an ensemble of spin$-1/2$ states in which the two eigenstates of σ_x appear with the same probabilities (equal of course to $1/2$). Now do the same for an equal mixture of eigenstates of $\hat{\sigma}_y$. And then consider an unequal mixture: an ensemble in which $|+x\rangle$ appears with probability $1/4$ and $|-x\rangle$ with probability $3/4$. In the “standard” basis the matrix is not diagonal- but you should be able to guess its eigenvalues. Finally, write down the density matrix corresponding to an ensemble in which $|+z\rangle$ appears with probability of $1/2$ and $|+x\rangle$ appears with the same probability. What are the eigenstates and eigenvalues of this density matrix?

Problem 2.6

Check that every density matrix $\hat{\rho}$ describing a states of a qubit can be written as

$$\hat{\rho} = \frac{1}{2}\left(\mathbb{I} + n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z\right) \ . \tag{14}$$

What is the relation between n_i and expectation values of $\hat{\sigma}_i$?

Bonus: a tricky question from Asher Peres’ book.

It should be obvious that if you are just given an ensemble of states to do experiments on, you cannot discover in what way this ensemble was prepared – the two first “ensemble preparation recipes” from problem 2.5 correspond to *the same* density matrix, and this matrix determines all the possible expectation values and statistical distibutions of experimental results. It is however enlightening to consider a situation in which we are given some partial information about the way in which the ensemble was prepared. Let us assume that we are given a large number N of spins, and we are told that we either got $N/2$ spins in $|+z\rangle$ state and $N/2$ spins in $|-z\rangle$ state, or we got $N/2$ spins in $|+x\rangle$ state and $N/2$ spins in $|-x\rangle$ state. Can we do now measurements on each of the members of this ensemble that will allow us to tell *with high probability* which of the two ensemble preparation procedures had been in fact used?