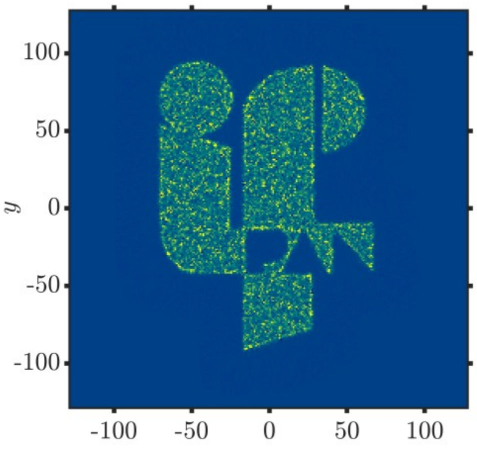
Simulating the complete quantum mechanics of very large dissipative Bose-Hubbard models



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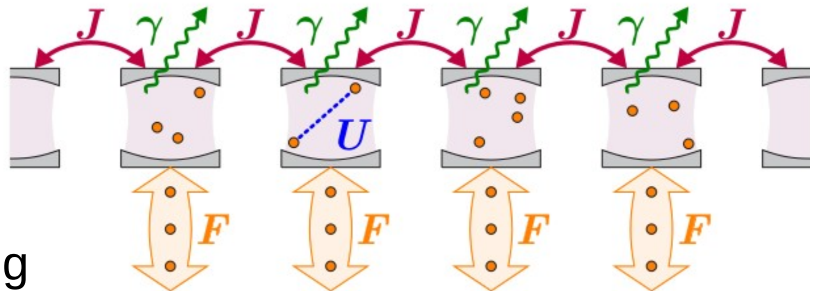
References:

- PD, Ferrier, Matuszewski, Orso, Szymańska, PRX Quantum **2**, 010319 (2021).
- PD, Quantum **5**, 455 (2021).

Outline

- Driven dissipative Bose-Hubbard model
- About quantum complexity and phase-space representations
  - the *positive-P* method
  - The *truncated Wigner* method
- Simulations of the driven-dissipative Bose-Hubbard model
- Outlook

$$\widehat{H} = \sum_j \widehat{H}_j - \sum_{\text{connections } i,j} \left[J_{ij} \widehat{a}_j^\dagger \widehat{a}_i + J_{ij}^* \widehat{a}_i^\dagger \widehat{a}_j \right]$$



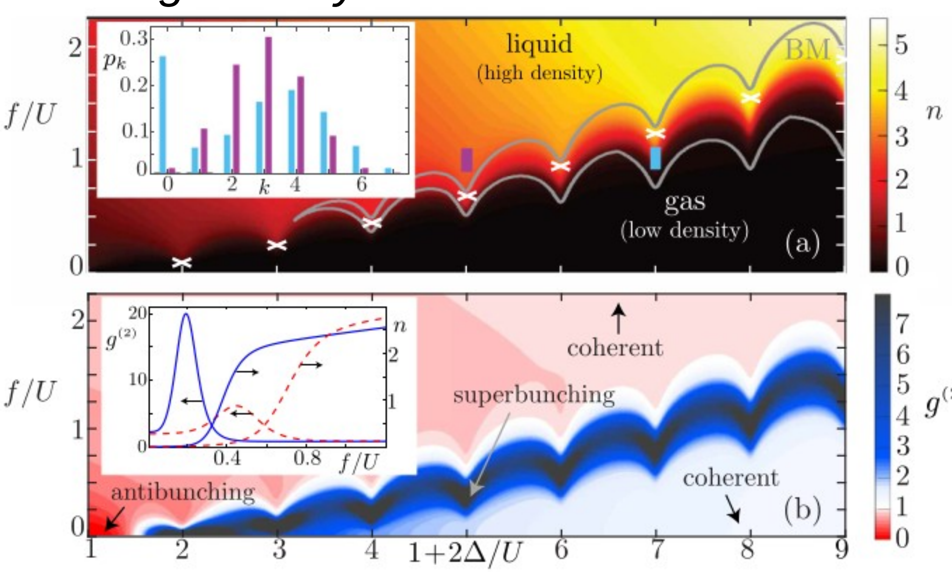
driving

$$\widehat{H}_j = -\Delta_j \widehat{a}_j^\dagger \widehat{a}_j + \frac{U_j}{2} \widehat{a}_j^\dagger \widehat{a}_j^\dagger \widehat{a}_j \widehat{a}_j + F_j \widehat{a}_j^\dagger + F_j^* \widehat{a}_j$$

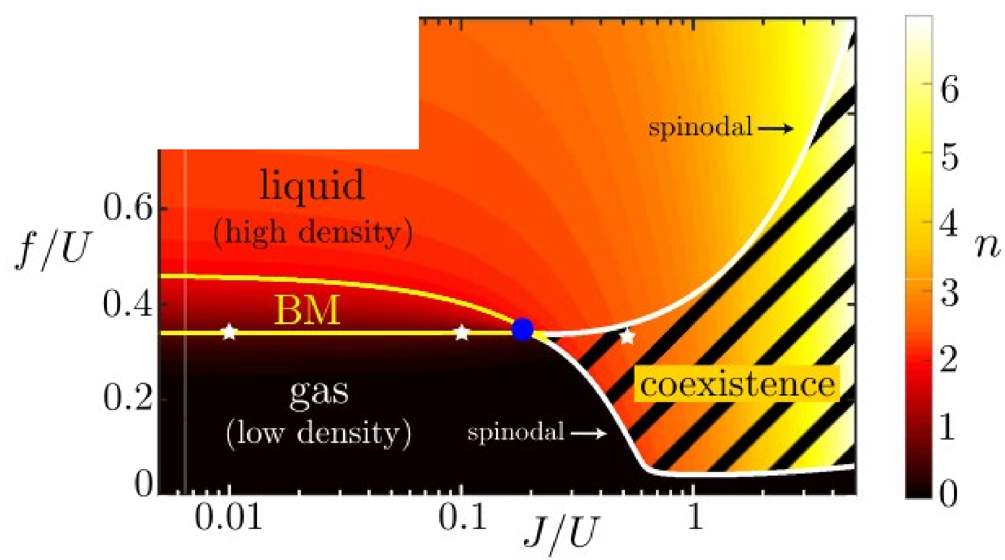
dissipation

$$\frac{\partial \widehat{\rho}}{\partial t} = -i \left[\widehat{H}, \widehat{\rho} \right] + \sum_j \frac{\gamma_j}{2} \left[2 \widehat{a}_j \widehat{\rho} \widehat{a}_j^\dagger - \widehat{a}_j^\dagger \widehat{a}_j \widehat{\rho} - \widehat{\rho} \widehat{a}_j^\dagger \widehat{a}_j \right]$$

single cavity



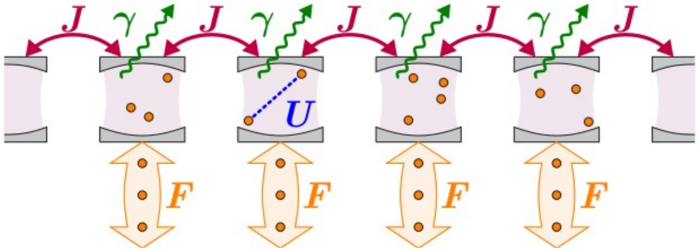
lattice, mean-field decoupling approx



Biondi, Blatter, Tureci, Schmidt, PRA **96**, 043809 (2017)

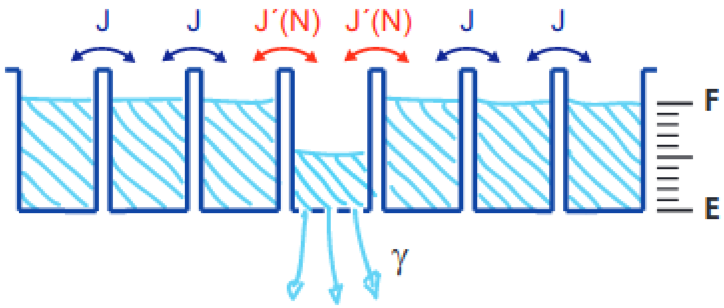
Examples of dissipative Bose-Hubbard systems

Arrays of optical cavities



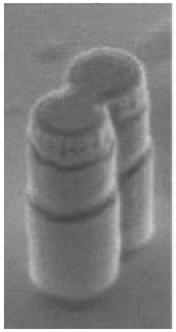
Vincentini, Minganti, Rota, Orso, Ciuti, PRA **97**, 013853 (2018)

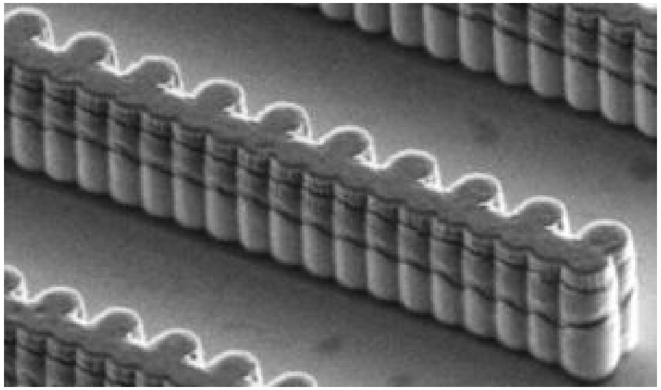
Ultracold atoms in optical lattice with forced losses



Labouvie, Santra, Heun, Ott, PRL **116**, 235302 (2016)

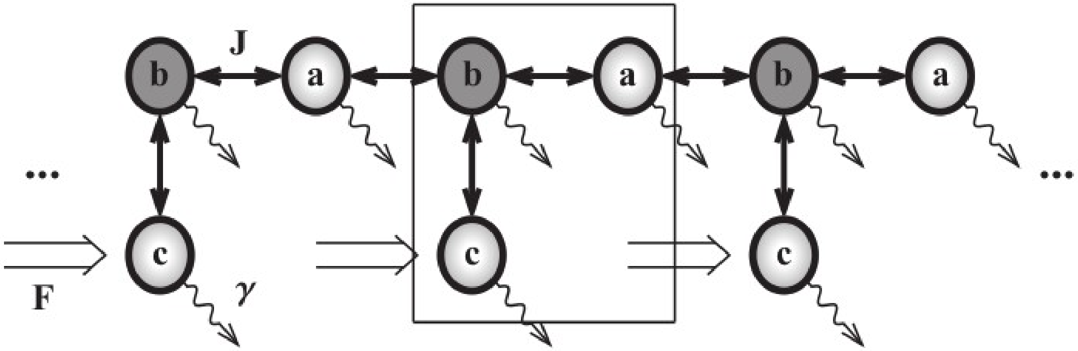
Polaritons in micropillars





Baboux, Ge, Jacqmin, Biondi, Galopin,
Lemaitre, Le Gratiet, Sagnes, Schmidt,
Tureci, Amo, Bloch, PRL **116**, 066402 (2016)

Also structured lattices – e.g. Lieb lattice



Casteels, Rota, Storme, Ciuti, PRA **93**, 043833 (2016)

Difficulties of simulating quantum mechanics

Suppose we have a system with M sites, each site can have 0,1,..., (d-1) particles

orbitals *(2,3,...,d-level systems)*

modes

How many variables do we need to describe the state?

- Classical physics: configuration $\{n_1, n_2, \ldots, n_M\}$ $\sim M$ real variables
- Closed quantum system: state vector $\sim d^M$ complex variables

$$|\Psi\rangle = \sum_{n_1=0}^{d-1} \sum_{n_2=0}^{d-1} \cdots \sum_{n_M=0}^{d-1} C_{n_1,n_2,\cdots,n_M} |n_1\rangle \otimes |n_2\rangle \cdots \otimes |n_M\rangle$$

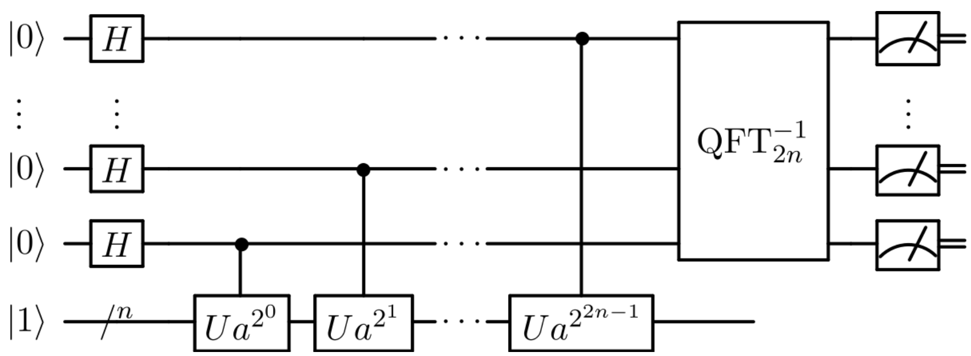
- Open quantum system: density matrix $\sim \frac{1}{2} d^{2M}$ complex variables

$$\hat{\rho} = \sum_j \lambda_j |\Psi_j\rangle\langle\Psi_j|$$

How can there be hope?

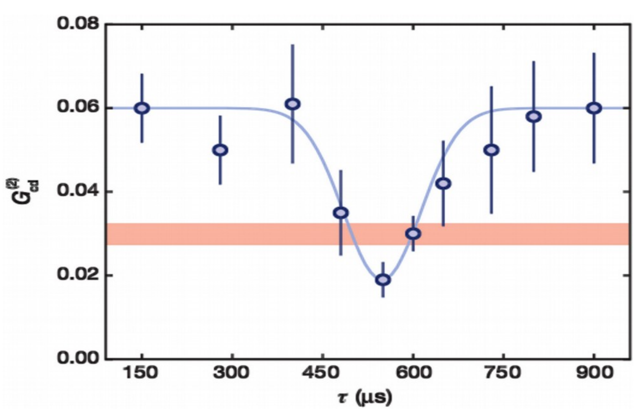
(Type 1) Universal quantum computer, Shor's, Grover's algorithms, etc...

* needs precise knowledge of microscopic subsystem observables



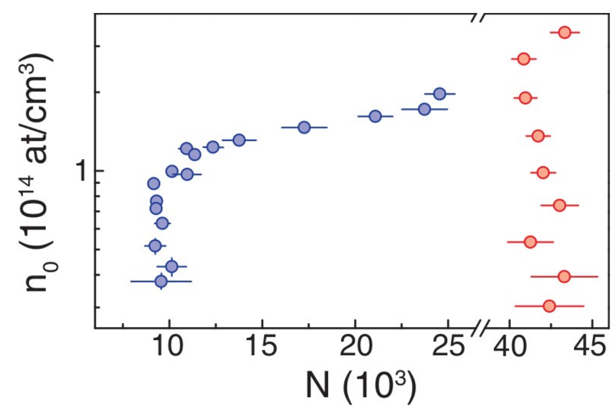
(Type 2) Quantum behaviour of (most?) experimental systems

* knowledge of bulk or locally averaged quantities suffices
* statistical uncertainty mirrors experimental reality



Lopes, Imanaliev, Aspect, Cheneau, Boiron, Westbrook, *Nature* **520**, 66 (2015)

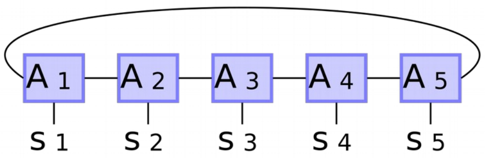
Cabrera, Tanzi, Sanz, Naylor, Thomas, Cheiney, Tarruell, *Science* **359**, 301 (2018)

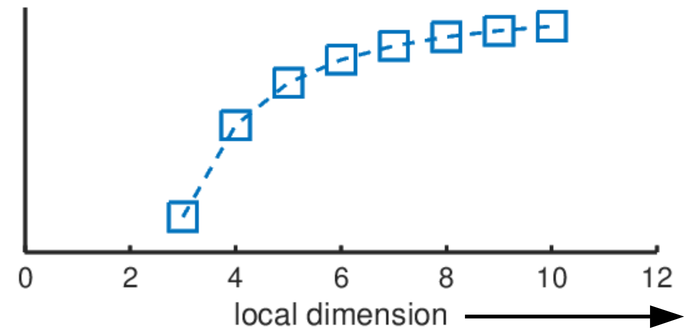


Complete quantum mechanics with limited precision

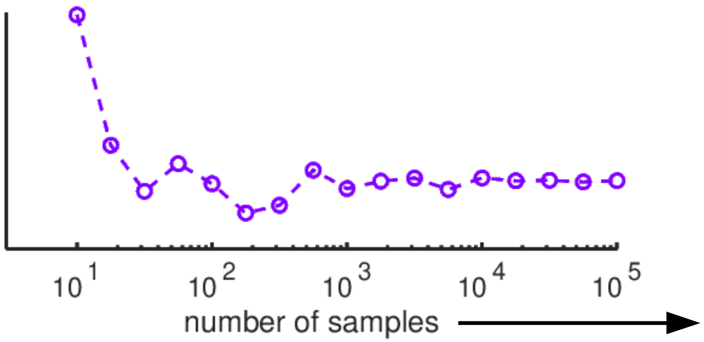
Approach full quantum predictions as some technical parameter is changed

- "Asymptotic approach"
  - *Reduce basis set but treat what happens inside exactly*
  - matrix product states, "DMRG", PEPS
  - multi-configuration methods, MCDHF...
  - ....





- "Stochastic approach"
  - *Use full basis but randomly choose a few configurations*
  - path integral Monte Carlo
  - phase space methods (positive-P, Wigner...)
  - ...



- Each approach has its own quirky pros and cons

Positive-P representation 1/3: configurations

M subsystems (modes, sites, volumes) labeled by j

Coherent state basis, complex, *local* α_j

$$|\alpha_j\rangle_j = e^{-|\alpha_j|^2/2}\, e^{\alpha_j \widehat{a}_j^\dagger} |\text{vac}\rangle$$

"ket" amplitude α_j
"bra" amplitude β_j^*

Local operator kernel

$$\widehat{\Lambda}(\boldsymbol{\lambda}) = \bigotimes_j \frac{|\alpha_j\rangle_j \langle \beta_j^*|_j}{\langle \beta_j^*|_j |\alpha_j\rangle_j}$$

full system configuration

$$\boldsymbol{\lambda} = \{\alpha_1, \dots, \alpha_M, \beta_1, \dots, \beta_M\}$$

Full density matrix

$$\widehat{\rho} = \int d^{4M}\boldsymbol{\lambda}\; P_+(\boldsymbol{\lambda})\, \widehat{\Lambda}(\boldsymbol{\lambda})$$

Correlations between subsystems are all in the distribution of configurations

$P_+(\boldsymbol{\lambda})$ The distribution is positive, real ⟶ let's SAMPLE IT !

Crucial element: differential identities

Follow from the operator kernel:

$$\widehat{\Lambda}(\boldsymbol{\lambda}) = \bigotimes_j \frac{|\alpha_j\rangle_j \langle\beta_j^*|_j}{\langle\beta_j^*|_j|\alpha_j\rangle_j}$$

$$\widehat{a}_j\widehat{\Lambda} = \alpha_j\widehat{\Lambda},$$
$$\widehat{a}_j^\dagger\widehat{\Lambda} = \left[\beta_j + \frac{\partial}{\partial\alpha_j}\right]\widehat{\Lambda}$$
$$\widehat{\Lambda}\widehat{a}_j = \left[\alpha_j + \frac{\partial}{\partial\beta_j}\right]\widehat{\Lambda}$$
$$\widehat{\Lambda}\widehat{a}_j^\dagger = \beta_j\widehat{\Lambda}.$$

Observable: occupation

$$\langle\widehat{N}_j\rangle = \langle\widehat{a}_j^\dagger\widehat{a}_j\rangle = \mathrm{Tr}\left[\widehat{a}_j^\dagger\widehat{a}_j\widehat{\rho}\right]$$

$$\widehat{\rho} = \int d^{4M}\boldsymbol{\lambda}\, P_+(\boldsymbol{\lambda})\,\widehat{\Lambda}(\boldsymbol{\lambda})$$

$$= \int d^{4M}\boldsymbol{\lambda}\, P_+(\boldsymbol{\lambda})\;\mathrm{Tr}\left[\widehat{a}_j^\dagger\widehat{a}_j\widehat{\rho}\right]$$

$$= \int d^{4M}\boldsymbol{\lambda}\, P_+(\boldsymbol{\lambda})\;\alpha_j\left[\beta_j + \frac{\partial}{\partial\alpha_j}\right]\mathrm{Tr}\left[\widehat{\Lambda}\right]$$

$$\mathrm{Tr}\left[\widehat{\Lambda}\right] = 1$$

$$= \int d^{4M}\boldsymbol{\lambda}\, P_+(\boldsymbol{\lambda})\;\alpha_j\beta_j$$

$$= \lim_{\mathcal{S}\to\infty}\langle\alpha_j\beta_j\rangle_{\mathrm{stoch}}.$$

$$\widehat{\rho} = \int d^{4M}\boldsymbol{\lambda}\, P_+(\boldsymbol{\lambda})\,\widehat{\Lambda}(\boldsymbol{\lambda})$$

we have *S* samples of $\boldsymbol{\lambda}$
the configurations
distributed according to $P_+(\boldsymbol{\lambda})$

NARODOWE CENTRUM NAUKI

Density matrix $\hat{\rho}$ ↔ distribution P_+ for the fields ↔ random samples of the fields α β

Master equation:

dissipation γ

$$\frac{\partial\hat{\rho}}{\partial t} = -i\left[\hat{H},\hat{\rho}\right] + \frac{\gamma}{2}\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right)$$

Bose-Hubbard site

$$\hat{H} = \frac{U}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} - \Delta\hat{a}^\dagger\hat{a}$$

apply identities

Fokker Planck equation

$$\frac{\partial P_+}{\partial t} = \left\{-\frac{\partial}{\partial\alpha}\left(-iU\alpha\beta + i\Delta - \frac{\gamma}{2}\right)\alpha - \frac{\partial}{\partial\beta}\left(iU\alpha\beta - i\Delta - \frac{\gamma}{2}\right)\beta + \frac{\partial^2}{\partial\alpha^2}\left(\frac{-iU}{2}\right)\alpha^2 + \frac{\partial^2}{\partial\beta^2}\left(\frac{iU}{2}\right)\beta^2\right\}P_+$$

deterministic GPE (ket) — deterministic (bra) — diffusion (quantum noise)

stochastic correspondence

Stochastic (Langevin) equations:

different noises

$$\frac{d\alpha}{dt} = \left(-iU\alpha\beta + i\Delta - \frac{\gamma}{2}\right)\alpha + \sqrt{-iU}\,\alpha\xi(t)$$

$$\frac{d\beta}{dt} = \left(+iU\alpha\beta - i\Delta - \frac{\gamma}{2}\right)\beta + \sqrt{+iU}\,\beta\widetilde{\xi}(t)$$

mean field part — quantum noise part

White noise

$$\langle\xi(t)\xi(t')\rangle_{\text{stoch}} = \delta(t-t')$$

Overall properties for a many-body system

Density matrix $\widehat{\rho}$ ↔ distribution P_+ for the fields ↔ random samples of the fields α β

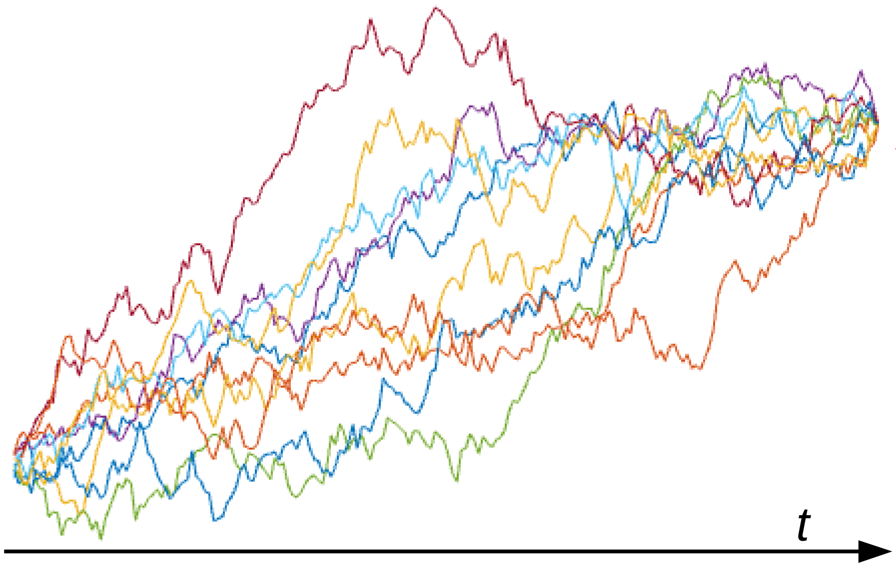
- From $\sim \frac{1}{2} d^{2M}$ variables we go to S samples x $2M$
- Therefore ---- **Scales extremely well (linear with the number of modes M)**
- Requires no particular symmetries, explicit time-dependence trivially added
- BUT **can be unstable**, usually no good for long-time equilibrium.

[more on this later !]

- Still..... particularly suited to large and "dirty" problems.

Dirty problems done dirt cheap!

Path integral Monte Carlo



Configuration size:

N particles x *T* time slices

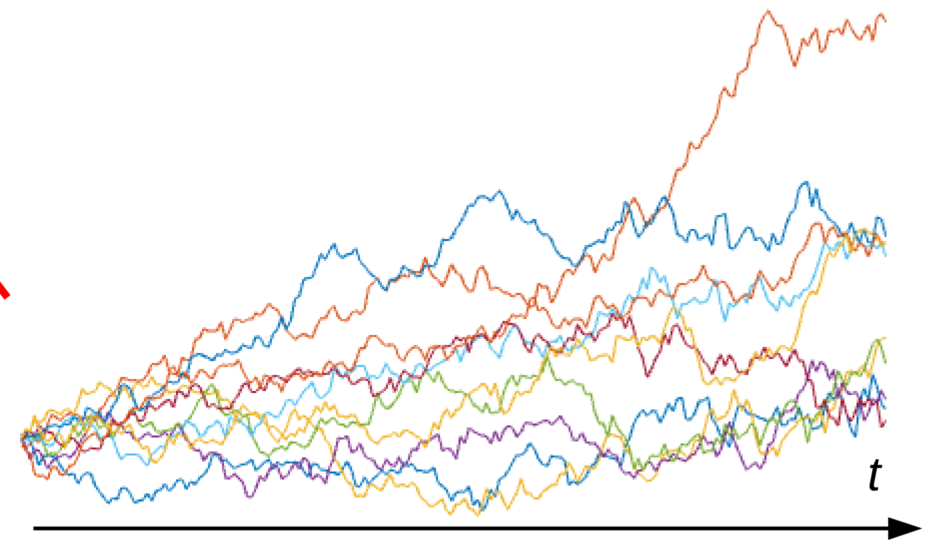
Typical algorithm: Metropolis

weights

--> phase problem in dynamics

no noise amplification problem

Phase-space representation



Configuration size:

M modes

Typical algorithm: Langevin equations

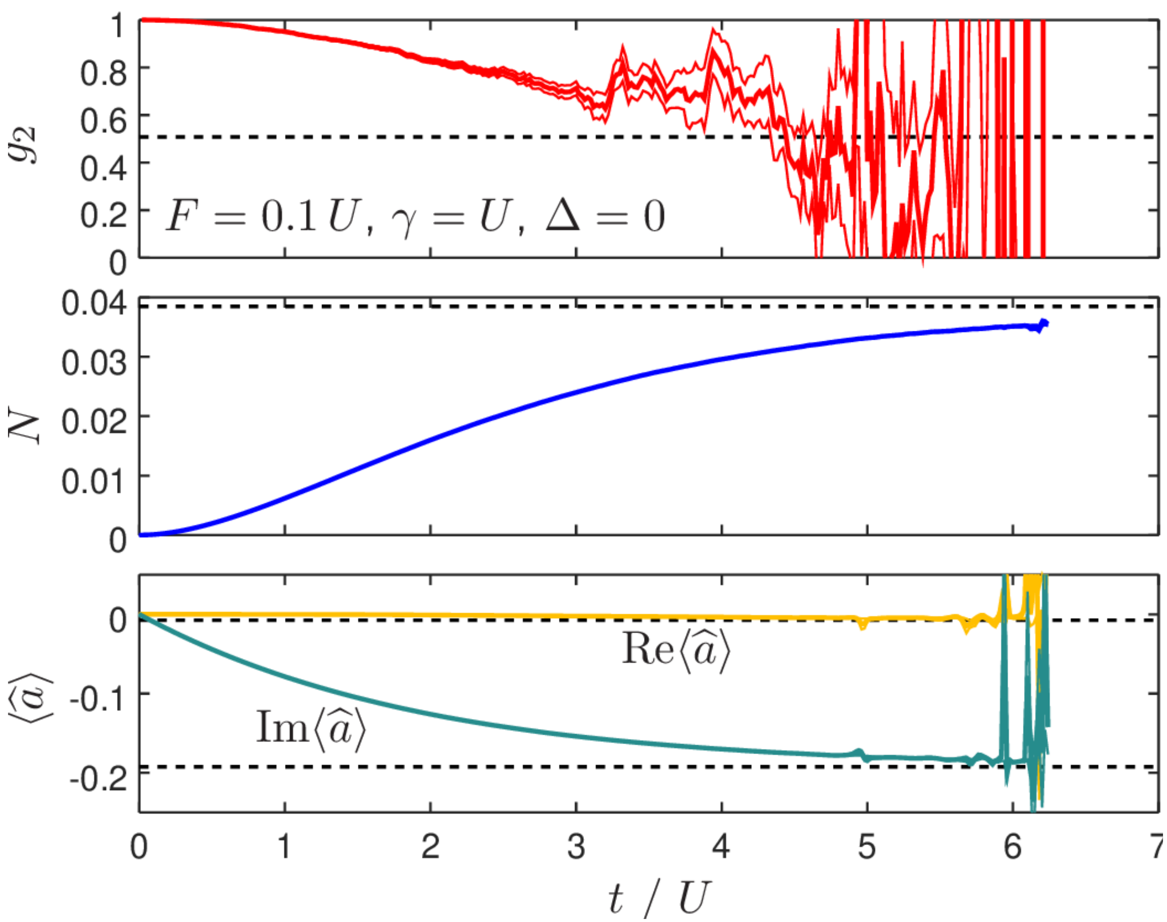
no weights for dynamics

--> no sign problem there

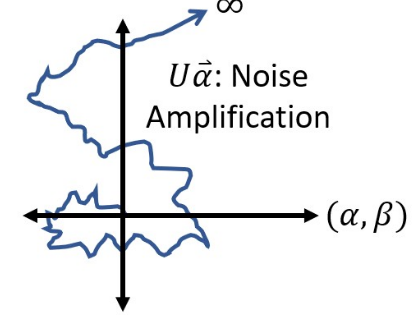
noise amplification if dissipation insufficient

Achilles heel – noise amplification limits simulation time

EXAMPLE



here be some
stupid location
in phase space



long time dynamics excluded

Particularly for closed Hamiltonian systems.

Dealing with noise amplification

- It was found that the simulation time is limited:

$$t_{\text{sim}} \approx \begin{cases} \dfrac{2.5}{\max_j [U_j N_j^{2/3}]} & \text{if } \max_j N_j \gg 1, \\ \dfrac{C}{\max_j U_j} & \text{if } \max_j N_j \ll 1, \end{cases}$$

- Various ways have been developed to improve this performance:
  - * stochastic Gauges — PD, Drummond, *PRA* **66**, 033812 (2002), J Phys A **39**, 2723 (2006); PD *et al*, PRA **79**, 043619 (2009); Wuster, Corney, Rost, PD, PRE **96**, 013309 (2017)
  - * quantum interpolation — PD, *PRL* **103,** 130402 (2009); Ng, Sorensen, PD, PRB **88**, 144304 (2013)
- Or it can be optimal to just use approximate representations:
  - * truncated Wigner — Sinatra, Lobo, Castin, J Phys B 35, 3599 (2002) Norrie, Ballagh, Gardiner, *PRA* **73**, 043617 (2006), PRL **94**, 040401 (2005)
  - * STAB (Stochastic adaptive Bogoliubov) — PD, Chwedeńczuk, Trippenbach, Zin, PRA 83, 063625 (2011) Kheruntsyan *et al*, *PRL* **108**, 260401 (2012)

- It was also found that simulation time grows with dissipation to an external bath:

$$t_{\text{sim}} \sim \frac{2 - \log N}{U - \gamma}$$

- But not really tested at the time ....

positive-P representation of the driven dissipative BH model

PD, Ferrier, Matuszewski, Orso, Szymańska, PRX Quantum **2**, 010319 (2021).

- Evolution equations for samples

$$\frac{\partial \alpha_j}{\partial t} = i\Delta_j \alpha_j - iU_j \alpha_j^2 \widetilde{\alpha}_j^* - iF_j - \frac{\gamma_j}{2}\alpha_j + \sqrt{-iU_j}\,\alpha_j\,\xi_j(t) + \sum_k iJ_{kj}\alpha_k,$$

$$\frac{\partial \widetilde{\alpha}_j}{\partial t} = i\Delta_j \widetilde{\alpha}_j - iU_j \widetilde{\alpha}_j^2 \alpha_j^* - iF_j - \frac{\gamma_j}{2}\widetilde{\alpha}_j + \sqrt{-iU_j}\,\widetilde{\alpha}_j\,\widetilde{\xi}_j(t) + \sum_k iJ_{kj}\widetilde{\alpha}_k$$

White Gaussian noise deals with interparticle collisions

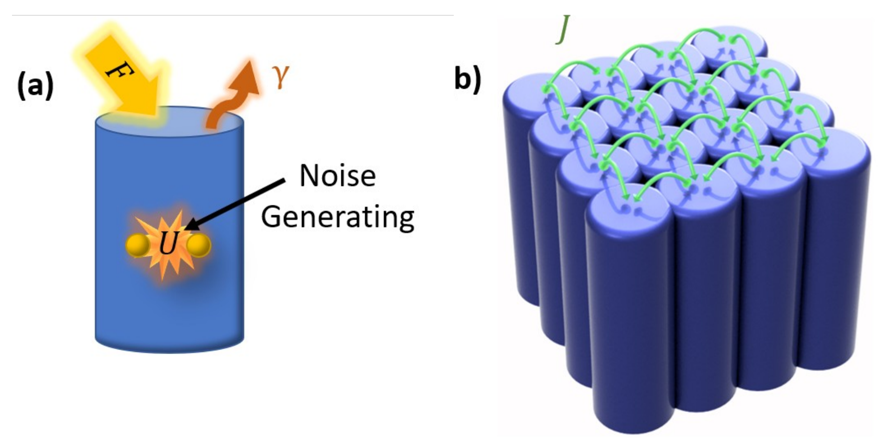
$$\langle \xi_j(t)\xi_k(t')\rangle_s = \delta(t-t')\delta_{jk}, \quad \langle \widetilde{\xi}_j(t)\widetilde{\xi}_k(t')\rangle_s = \delta(t-t')\delta_{jk}$$

The rest of the equations is basically mean field

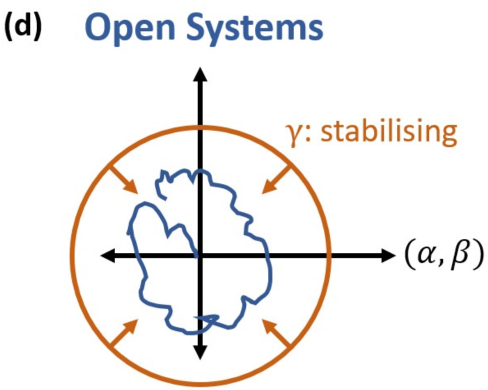
$$\widehat{H} = \sum_j \widehat{H}_j - \sum_{\text{connections}\, i,j} \left[J_{ij}\widehat{a}_j^\dagger \widehat{a}_i + J_{ij}^* \widehat{a}_i^\dagger \widehat{a}_j\right]$$

$$\widehat{H}_j = -\Delta_j \widehat{a}_j^\dagger \widehat{a}_j + \frac{U_j}{2}\widehat{a}_j^\dagger \widehat{a}_j^\dagger \widehat{a}_j \widehat{a}_j + F_j \widehat{a}_j^\dagger + F_j^* \widehat{a}_j$$

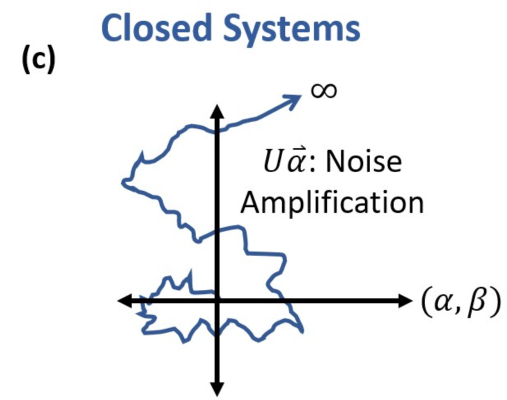
$$\frac{\partial \widehat{\rho}}{\partial t} = -i\left[\widehat{H}, \widehat{\rho}\right] + \sum_j \frac{\gamma_j}{2}\left[2\widehat{a}_j \widehat{\rho}\, \widehat{a}_j^\dagger - \widehat{a}_j^\dagger \widehat{a}_j \widehat{\rho} - \widehat{\rho}\, \widehat{a}_j^\dagger \widehat{a}_j\right]$$

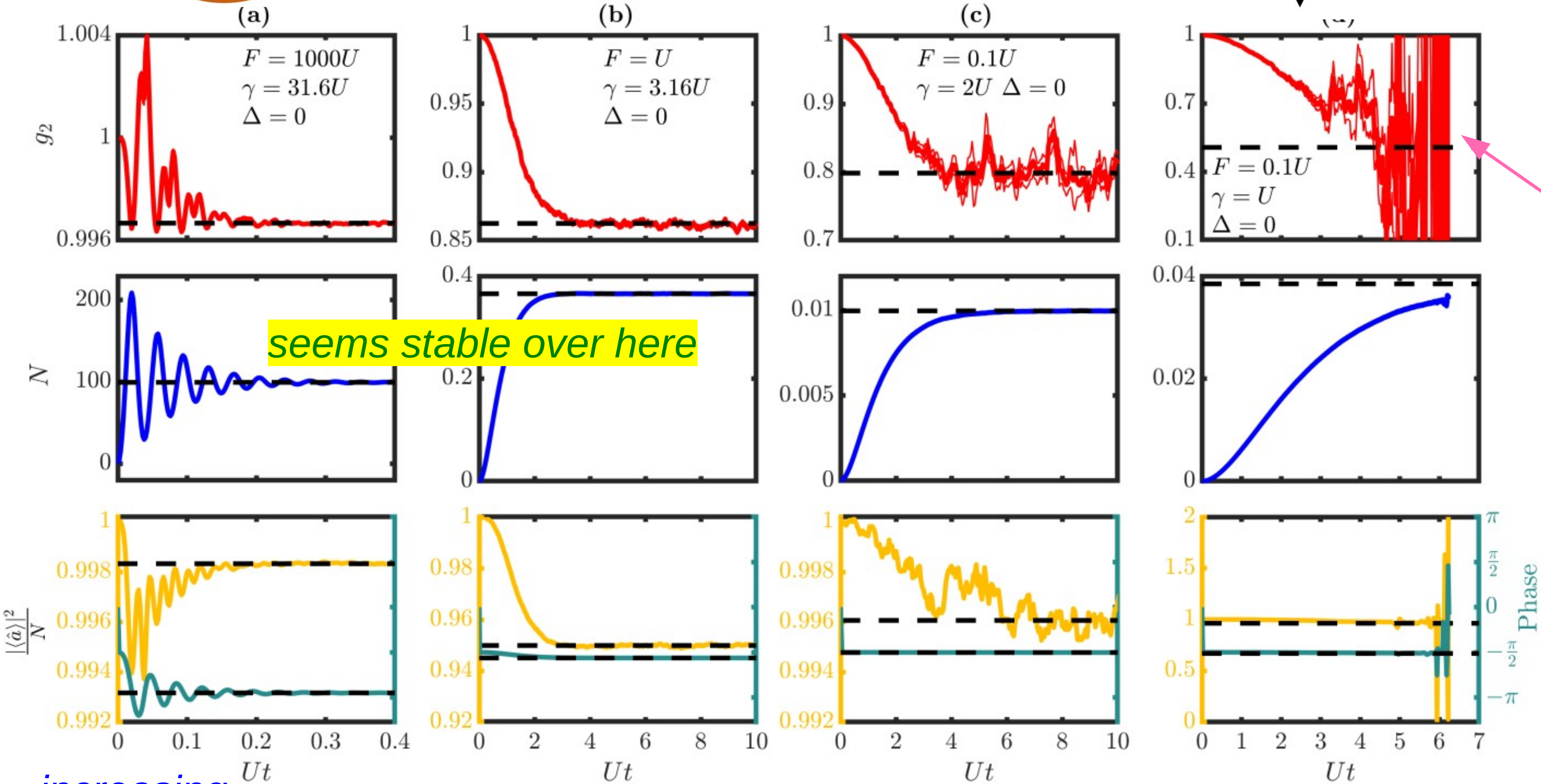


Positive-P simulations stabilised by the dissipation



stationary state can be reached
and studied
in a full quantum description





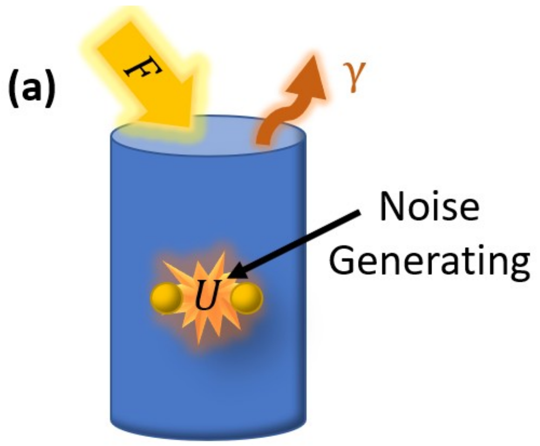
instability triggered γ too low

increasing dissipation

single Bose-Hubard site

Regime of stability for the positive-P approach

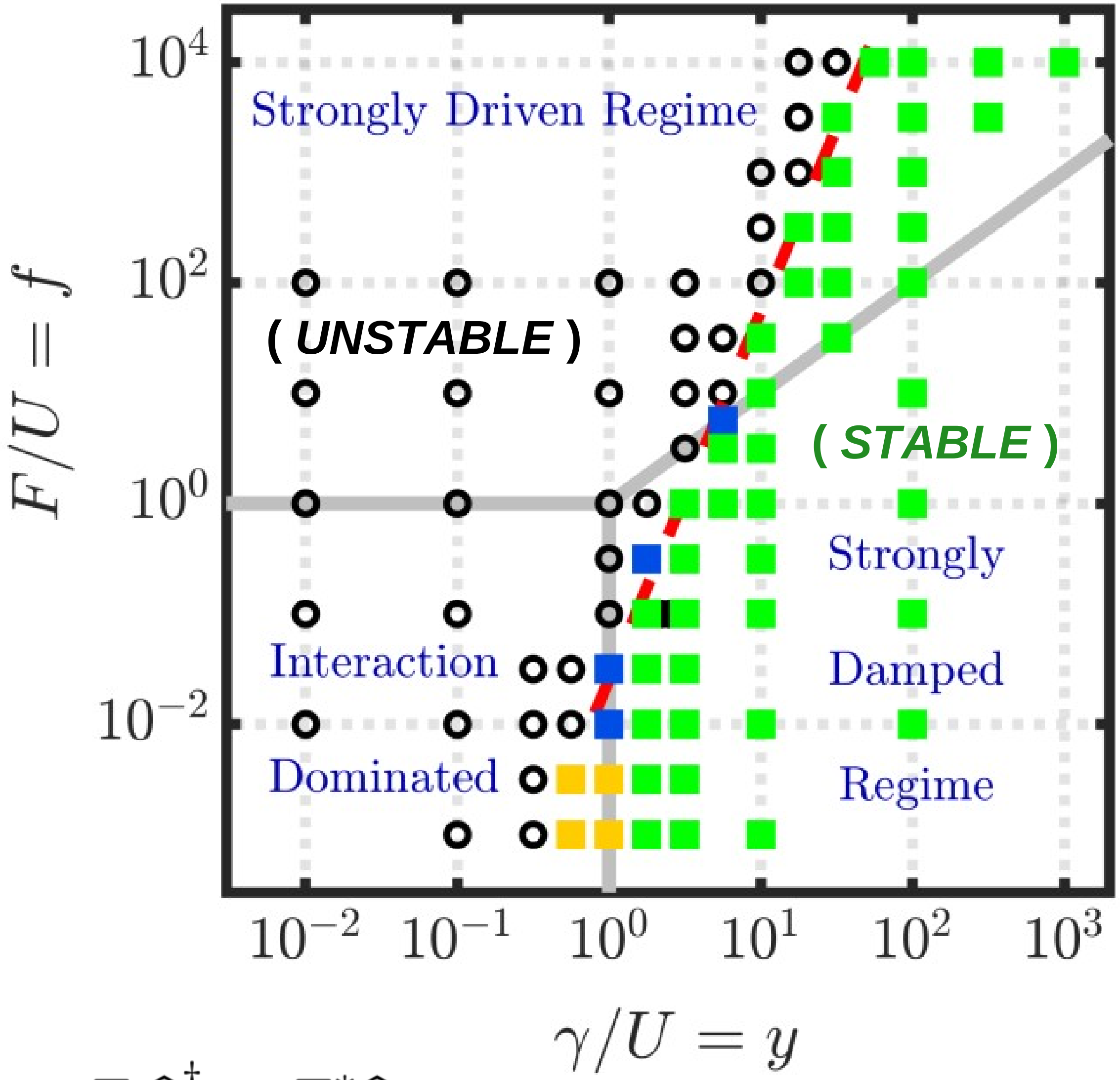
- Remarkably, stability is determined by single-site parameters



Regime of stability:

$$\gamma \gtrsim 3U\left(\frac{F}{U}\right)^{0.30}$$

$$\gamma \gtrsim U \quad \text{when} \quad F \lesssim 0.01U$$



$$\widehat{H}_j = -\Delta_j \widehat{a}_j^\dagger \widehat{a}_j + \frac{U_j}{2}\widehat{a}_j^\dagger \widehat{a}_j^\dagger \widehat{a}_j \widehat{a}_j + F_j \widehat{a}_j^\dagger + F_j^* \widehat{a}_j$$

PD, Ferrier, Matuszewski, Orso, Szymańska, PRX Quantum **2**, 010319 (2021).

Truncated Wigner representation

Different operator basis, different operator identities

$$\hat{a}_j\hat{\Lambda}_W = \left[\alpha_j - \frac{1}{2}\frac{\partial}{\partial\alpha_j^*}\right]\hat{\Lambda}_W$$

$$\hat{a}_j^\dagger\hat{\Lambda}_W = \left[\alpha_j^* + \frac{1}{2}\frac{\partial}{\partial\alpha_j}\right]\hat{\Lambda}_W$$

$$\hat{\Lambda}_W\hat{a}_j = \left[\alpha_j + \frac{1}{2}\frac{\partial}{\partial\alpha_j^*}\right]\hat{\Lambda}_W$$

$$\hat{\Lambda}_W\hat{a}_j^\dagger = \left[\alpha_j^* - \frac{1}{2}\frac{\partial}{\partial\alpha_j}\right]\hat{\Lambda}_W$$

positive-P

$$\hat{a}_j\hat{\Lambda} = \alpha_j\hat{\Lambda},$$

$$\hat{a}_j^\dagger\hat{\Lambda} = \left[\beta_j + \frac{\partial}{\partial\alpha_j}\right]\hat{\Lambda}$$

$$\hat{\Lambda}\hat{a}_j = \left[\alpha_j + \frac{\partial}{\partial\beta_j}\right]\hat{\Lambda}$$

$$\hat{\Lambda}\hat{a}_j^\dagger = \beta_j\hat{\Lambda}.$$

$$\hat{H} = \frac{U}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} - \Delta\hat{a}^\dagger\hat{a}$$

$$\frac{\partial\hat{\rho}}{\partial t} = -i\left[\hat{H},\hat{\rho}\right] + \frac{\gamma}{2}\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right)$$

3rd order terms → need to be *truncated*

Different Fokker-Planck equation

$$\frac{\partial P_W}{\partial t} = \left\{-\frac{\partial}{\partial\alpha}\left(-iU\left(|\alpha|^2 - 1\right) + i\Delta - \frac{\gamma}{2}\right)\alpha + iU\frac{\partial^3}{\partial\alpha^*\partial\alpha^2}\frac{\alpha}{2} + \text{c.c.} + \frac{\gamma}{2}\frac{\partial^2}{\partial\alpha^*\partial\alpha}\right\}P_W$$

Deterministic GPE-1

diffusion (vacuum noise)

positive-P

$$\frac{\partial P_+}{\partial t} = \left\{-\frac{\partial}{\partial\alpha}\left(-iU\alpha\beta + i\Delta - \frac{\gamma}{2}\right)\alpha - \frac{\partial}{\partial\beta}\left(iU\alpha\beta - i\Delta - \frac{\gamma}{2}\right)\beta + \frac{\partial^2}{\partial\alpha^2}\left(\frac{-iU}{2}\right)\alpha^2 + \frac{\partial^2}{\partial\beta^2}\left(\frac{iU}{2}\right)\beta^2\right\}P_+$$

Truncated Wigner equations

Stochastic (Langevin) equations:

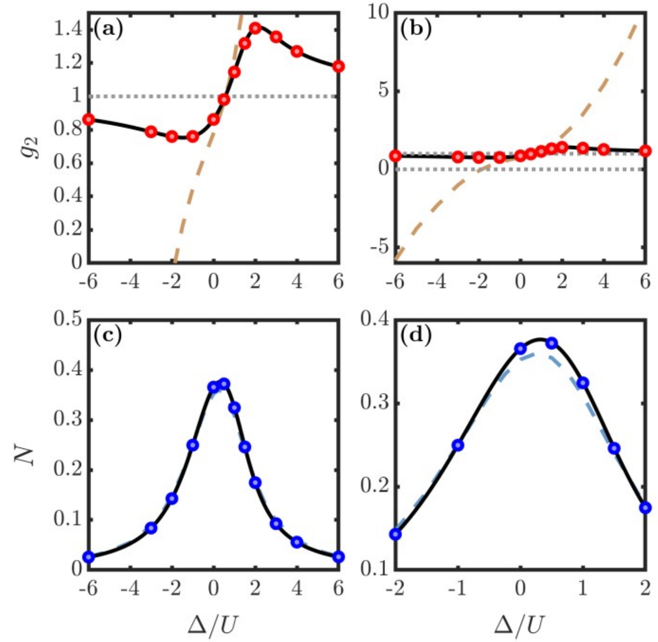
$$\frac{d\alpha}{dt} = \left(-iU\left(|\alpha|^2 - 1\right) + i\Delta - \frac{\gamma}{2}\right)\alpha + \sqrt{\frac{\gamma}{2}}\,\eta(t)$$

Initial condition:

$$\alpha_j(0) = \phi_j^0 + \frac{1}{\sqrt{2}}\eta_j \qquad \langle \eta_j^* \eta_k \rangle = \delta_{jk}$$

Always stable, sometimes not exact

$F = U, \gamma = 3.16U$



positive-P: symbols

trunc-Wigner: dashed

exact: solid

positive-P

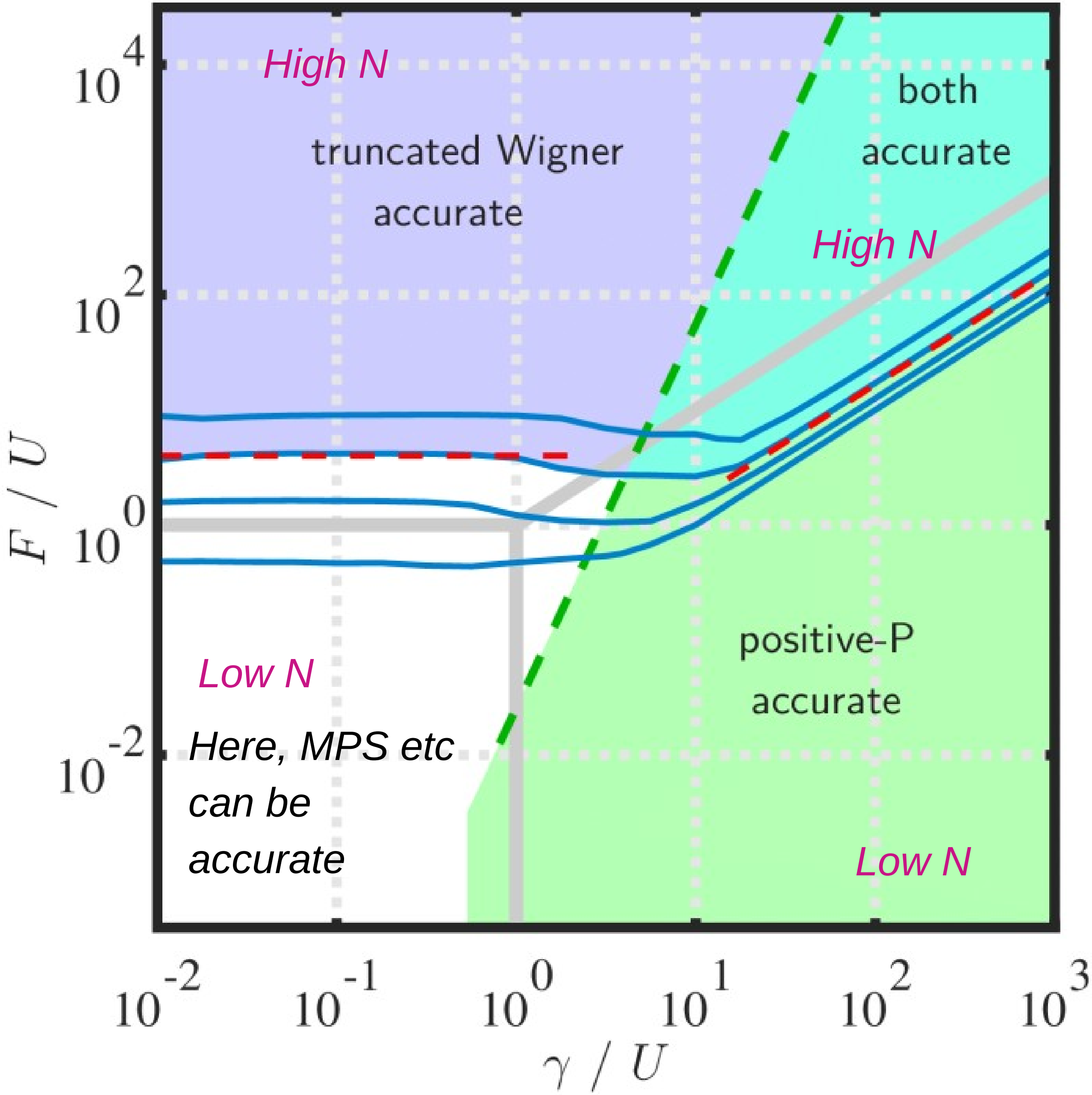
Exact but can be unstable

$$\alpha_j(0) = \phi_j^0 = \beta_j^*(0)$$

$$\frac{d\alpha}{dt} = \left(-iU\alpha\beta + i\Delta - \frac{\gamma}{2}\right)\alpha + \sqrt{-iU}\,\alpha\xi(t)$$

$$\frac{d\beta}{dt} = \left(+iU\alpha\beta - i\Delta - \frac{\gamma}{2}\right)\beta + \sqrt{+iU}\,\beta\widetilde{\xi}(t)$$

Phase space methods - regimes of applicability



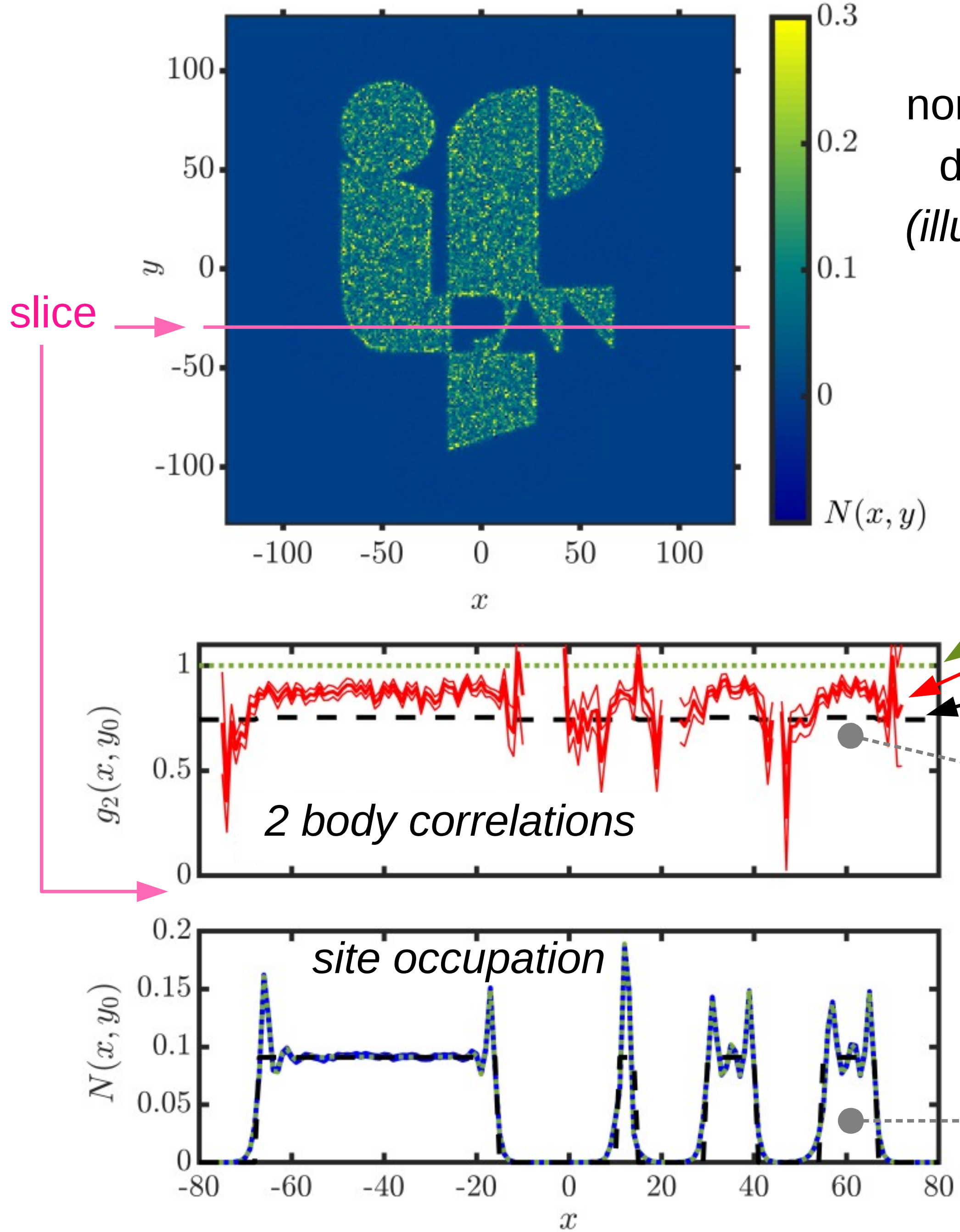
PD, Ferrier, Matuszewski, Orso, Szymańska, PRX Quantum **2**, 010319 (2021).

- Stability is largely independent of coupling *J*
- computational effort scales *linearly* with the size of the system.
- Computational effort is independent of dimensionality.
- Time-dependent system parameters and non-uniformity are trivial to implement.

pos-P Stability condition:

$$\gamma \gtrsim 3U \max\left[\sqrt{N}, 1\right]$$

Large systems – example simulation 256 x 256 lattice



non-uniform driving *F* *(illumination)*

$$\frac{1}{2}\,3^{2\times256\times256} \approx 10^{62\,000}$$

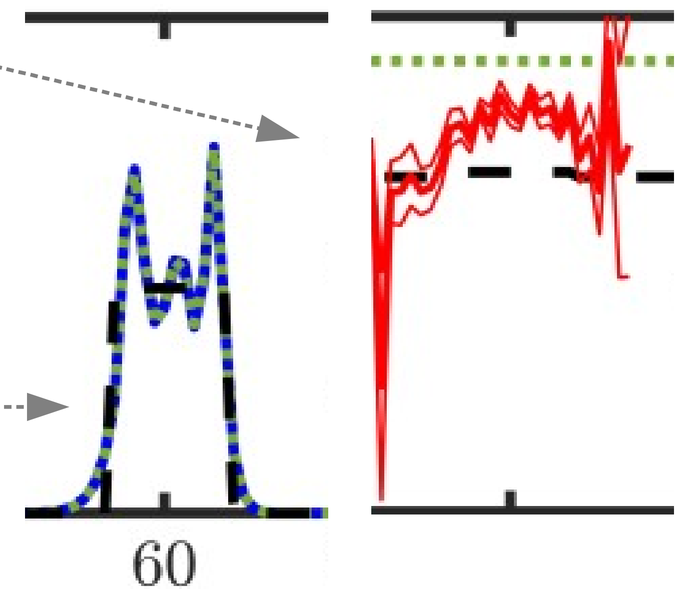
density matrix elements

GPE mean field

full quantum *(positive-P)*

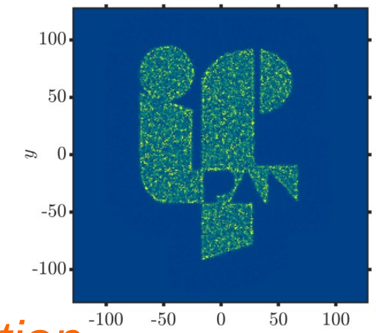
local density quantum model

LeBoite, Orso, Ciuti, PRL **110**, 233601 (2013).



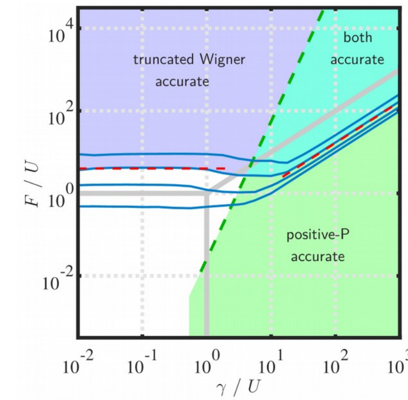
PD, Ferrier, Matuszewski, Orso, Szymańska, PRX Quantum **2**, 010319 (2021).

Summary and outlook



- Quantum dynamics of huge systems can be done
  under the right conditions, though stability issues

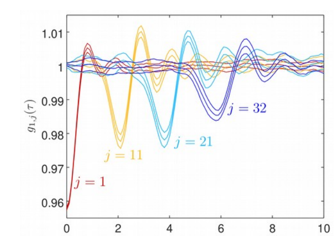
- Full quantum calculations of large Bose-Hubbard models
  positive-P method found to be stable with sufficient dissipation
  scalable. e.g. 10^5 sites is easy
  truncated Wigner accurate in complementary regimes



- Other dissipative models may also be possible.
  e.g. spins, Jaynes-Cummings-Hubbard
  Schwinger bosons — Ng, Sorensen, J Phys A **44**, 065305 (2011)
  Huber, Kirton, Rabl, SciPost Phys **10**, 045 (2021) (truncated Wigner)

  SU(n) positive-P-like representations — Ng, Sorensen, PD, PRB **88**, 144304 (2013)
  Begg, Green, Bhaseen arXiv:2011.07924 (stochastic gauges)

- Phase space approach also very good for multi-time correlations
  similar form to Heisenberg equations



References:

- PD, Ferrier, Matuszewski, Orso, Szymańska, *Fully Quantum Scalable Description of Driven-Dissipative Lattice Models*, PRX Quantum **2**, 010319 (2021).
- PD, *Multi-time correlations in the positive-P, Q, and doubled phase-space representations*, Quantum **5**, 455 (2021).

NARODOWE CENTRUM NAUKI

Unequal time correlations

- Expressed in terms of Heisenberg operators

$$\widehat{A}(t) = e^{i(t-t_0)\widehat{H}/\hbar}\ \widehat{A}(t_0)\ e^{-i(t-t_0)\widehat{H}/\hbar}$$

- Time ordered correlation functions.
  Correspond to all sequences of measurements

$$\langle \widehat{A}_1(t_1)\widehat{A}_2(t_2)\cdots\widehat{A}_{\mathcal{N}}(t_{\mathcal{N}})\widehat{B}_1(s_1)\widehat{B}_2(s_2)\cdots\widehat{B}_{\mathcal{M}}(s_{\mathcal{M}})\rangle \qquad \begin{array}{l} t_1 \leqslant t_2 \leqslant \ldots \leqslant t_{\mathcal{N}} \\ s_1 \geqslant s_2 \geqslant \ldots \geqslant s_{\mathcal{M}} \end{array}$$

time grows → ← time grows

- e.g.

$$\langle \widehat{a}^\dagger(0)\widehat{a}^\dagger(\tau)\widehat{a}(\tau)\widehat{a}(0)\rangle$$

particle present both at *t=0* and *t=tau*

$$\langle \widehat{a}^\dagger(\tau)\widehat{a}^\dagger(\tau)\widehat{a}(0)\widehat{a}(0)\rangle$$

anomalous pair correlation:
anihilate pair at *t=0* create at *t=tau*

Partial analogy positive-P <-> Heisenberg equations of motion

Correspondence in observable calculations:

$$\widehat{a} \leftrightarrow \alpha \qquad \widehat{a}^\dagger \leftrightarrow \beta$$

Heisenberg equations of motion:

$$\frac{d\,\widehat{a}(t)}{dt} = \left(-iU\widehat{a}^\dagger(t)\widehat{a}(t) + i\Delta - \frac{\gamma}{2}\right)\widehat{a}(t)$$
$$\frac{d\,\widehat{a}^\dagger(t)}{dt} = \widehat{a}^\dagger(t)\left(+iU\widehat{a}^\dagger(t)\widehat{a}(t) - i\Delta - \frac{\gamma}{2}\right)$$

$$\widehat{H} = \frac{U}{2}\widehat{a}^\dagger\widehat{a}^\dagger\widehat{a}\widehat{a} - \Delta\widehat{a}^\dagger\widehat{a}$$

positive-P equations of motion

$$\frac{d\alpha}{dt} = \left(-iU\alpha\beta + i\Delta - \frac{\gamma}{2}\right)\alpha + \sqrt{-iU}\,\alpha\xi(t)$$
$$\frac{d\beta}{dt} = \left(+iU\alpha\beta - i\Delta - \frac{\gamma}{2}\right)\beta + \sqrt{+iU}\,\beta\widetilde{\xi}(t)$$

Indeed, many unequal time correlations have remarkably simple expressions

$$g^{(2)}_{1,1}(\tau) = \frac{\langle\widehat{a}_1^\dagger(t)\,\widehat{a}_1^\dagger(t+\tau)\,\widehat{a}_1(t+\tau)\,\widehat{a}_1(t)\rangle}{\langle\widehat{a}_1^\dagger(t)\,\widehat{a}_1(t)\rangle\langle\widehat{a}_1^\dagger(t+\tau)\,\widehat{a}_1(t+\tau)\rangle} = \frac{\mathrm{Re}\langle\alpha_1(t)\,\alpha_1(t+\tau)\,\widetilde{\alpha}_1^*(t+\tau)\,\widetilde{\alpha}_1^*(t)\rangle_s}{N_1(t)\,N_1(t+\tau)}$$

PD Quantum **5**, 455 (2021).

Evaluation of variously ordered correlations

Normal ordering: positive-P variables

PD Quantum **5**, 455 (2021).

$$\langle \widehat{a}^\dagger_{p_1}(t_1)\cdots\widehat{a}^\dagger_{p_\mathcal{N}}(t_\mathcal{N})\widehat{a}_{q_1}(s_1)\cdots\widehat{a}_{q_\mathcal{M}}(s_\mathcal{M})\rangle$$
$$= \langle \beta_{p_1}(t_1)\cdots\beta_{p_\mathcal{N}}(t_\mathcal{N})\alpha_{q_1}(s_1)\cdots\alpha_{q_\mathcal{M}}(s_\mathcal{M})\rangle_\text{stoch}$$

Anti-normal ordering: Q distribution variables

$$\langle \widehat{a}_{p_1}(t_1)\cdots\widehat{a}_{p_\mathcal{N}}(t_\mathcal{N})\widehat{a}^\dagger_{q_1}(s_1)\cdots\widehat{a}^\dagger_{q_\mathcal{M}}(s_\mathcal{M})\rangle$$
$$= \langle \alpha'_{p_1}(t_1)\cdots\alpha'_{p_\mathcal{N}}(t_\mathcal{N})\beta'_{q_1}(s_1)\cdots\beta'_{q_\mathcal{M}}(s_\mathcal{M})\rangle_\text{stoch}$$

conversion P->Q

$$\alpha'_j = \alpha_j + \zeta_j \qquad ; \qquad \beta'_j = \beta_j + \zeta^*_j$$

$$\langle\zeta_j\rangle_\text{stoch} = 0\ ;\ \langle\zeta_j\zeta_k\rangle_\text{stoch} = 0\ ;\ \langle\zeta^*_j\zeta_k\rangle_\text{stoch} = 1$$

Mixed ordering:

1) sample what is possible using positive-P variables
2) convert variables to doubled-Q
3) sample what is possible using Q variables

Order (number of operators)	2nd order	3rd order	4th order
Total permutations	**12**	**56**	**240**
single time correlations	4	8	16
multi-time accessible with P representation	4	14	36
additional accessible with Q representation	4	14	36
additional accessible with mixed order (Sec. 5.4)	–	12	72
Total doable	**12**	**48**	**160**
time ordered not doable	–	–	–
Not time ordered, not doable	–	8	80

Table 2: A tally of $\widehat{a}$, $\widehat{a}^\dagger$ products involving up to four operators, evaluated at one of two times. The general form considered is $\langle\widehat{A}(t_a)\widehat{B}(t_b)\widehat{C}(t_c)\widehat{D}(t_d)\rangle$, where $\widehat{A},\widehat{B},\widehat{C},\widehat{D}$ can be either of $\widehat{a}$ or $\widehat{a}^\dagger$ (same mode), and the time arguments can take up to two distinct times $t=0$ and $t=\tau>0$.

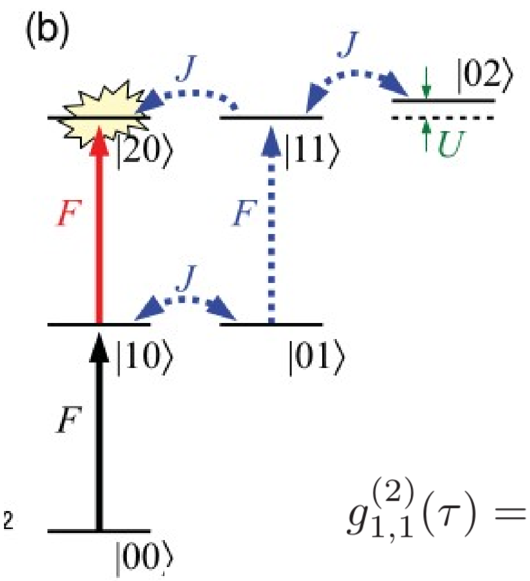
Unconventional photon blockade

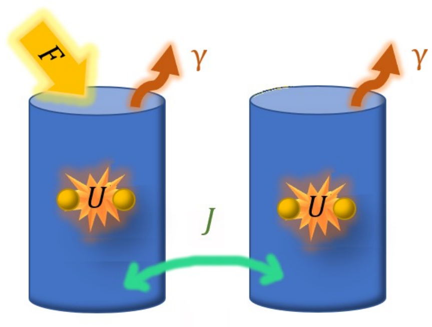
Complete antibunching,
Subtle interference effect $U \ll \gamma$

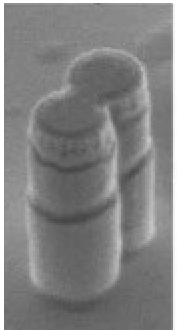
potential single-photon source

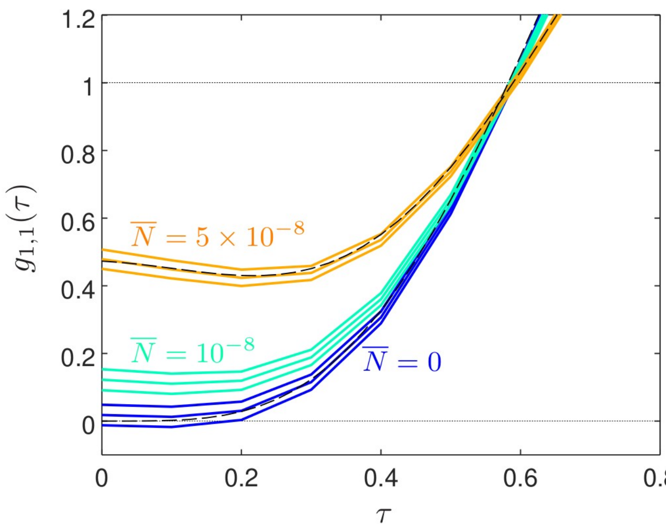
Liew, Savona, PRL **104**, 183601 (2010)
Bamba, Imamoglu, Carusotto, Ciuti, PRA **83**, 021802(R) (2011)

robustness to background photons:



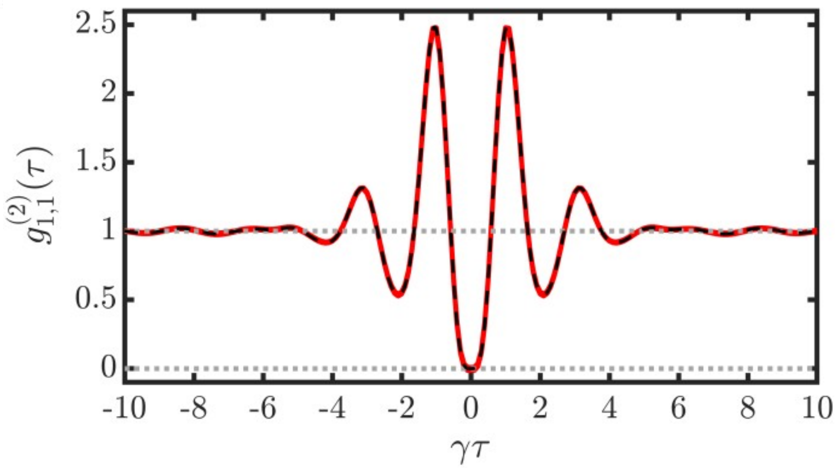






PD Quantum **5**, 455 (2021).

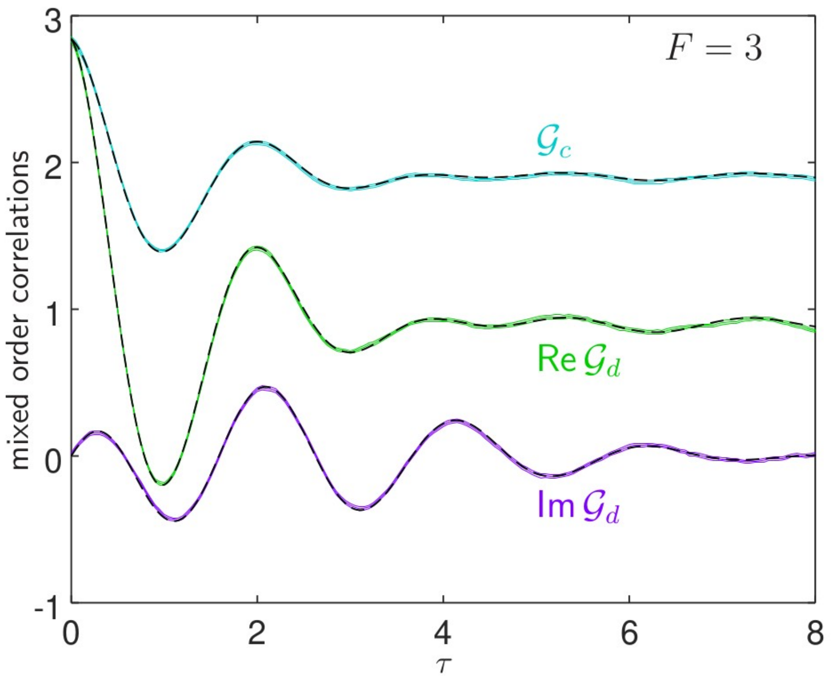
$$g^{(2)}_{1,1}(\tau) = \frac{1}{n^2} \langle \hat{a}^\dagger(0)\hat{a}^\dagger(\tau)\hat{a}(\tau)\hat{a}(0)\rangle$$



$$\frac{\partial\hat{\rho}}{\partial t} = -i\left[\hat{H},\hat{\rho}\right] + \frac{\gamma\overline{N}}{2}\sum_j\left[2\hat{a}_j^\dagger\hat{\rho}\hat{a}_j - \hat{a}_j\hat{a}_j^\dagger\hat{\rho} - \hat{\rho}^\dagger\hat{a}_j\hat{a}_j\right]$$
$$+\frac{\gamma(\overline{N}+1)}{2}\sum_j\left[2\hat{a}_j\hat{\rho}\hat{a}_j^\dagger - \hat{a}_j^\dagger\hat{a}_j\hat{\rho} - \hat{\rho}\hat{a}_j^\dagger\hat{a}_j\right]. \quad (96)$$

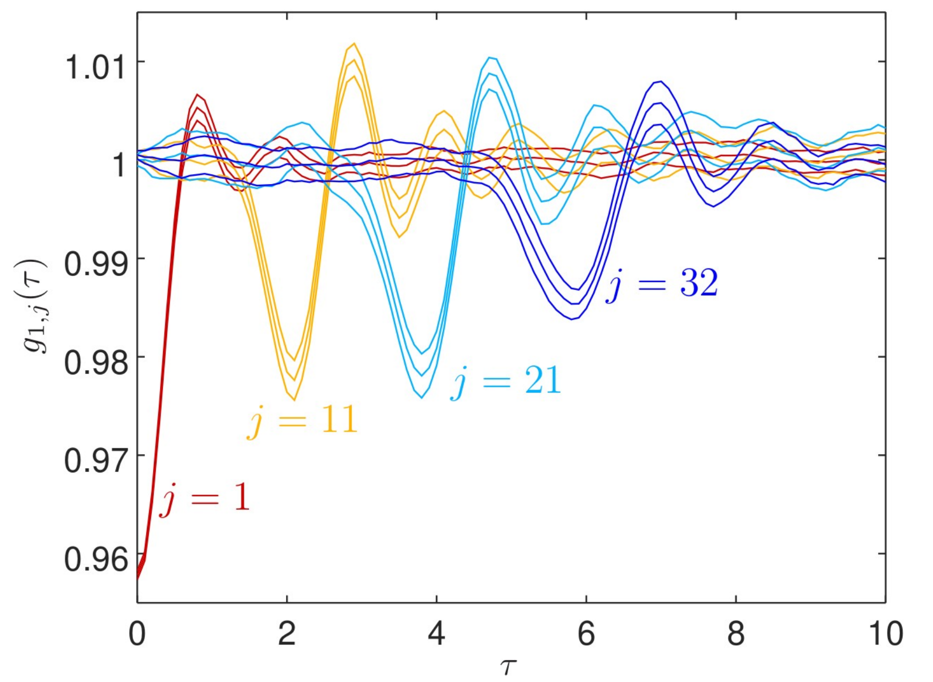
PD Quantum **5**, 455 (2021).

anomalously ordered correlations
by conversion to Q distribution



$$\mathcal{G}_c = \langle \widehat{a}_2(t_0+\tau)\widehat{a}_2^\dagger(t_0+\tau)\widehat{a}_2^\dagger(t_0)\widehat{a}_2(t_0)\rangle$$
$$= \mathrm{Re}\langle \alpha_2'(t_0+\tau)\beta_2'(t_0+\tau)\beta_2'(t_0)\alpha_2(t_0)\rangle_{\mathrm{stoch}}$$
$$\mathcal{G}_d = \langle \widehat{a}_2(t_0)[\widehat{a}_2^\dagger(t_0+\tau)]^2\widehat{a}_2(t_0)\rangle$$
$$= \langle \alpha_2'(t_0)\beta_2'(t_0+\tau)^2\alpha_2(t_0)\rangle_{\mathrm{stoch}}$$

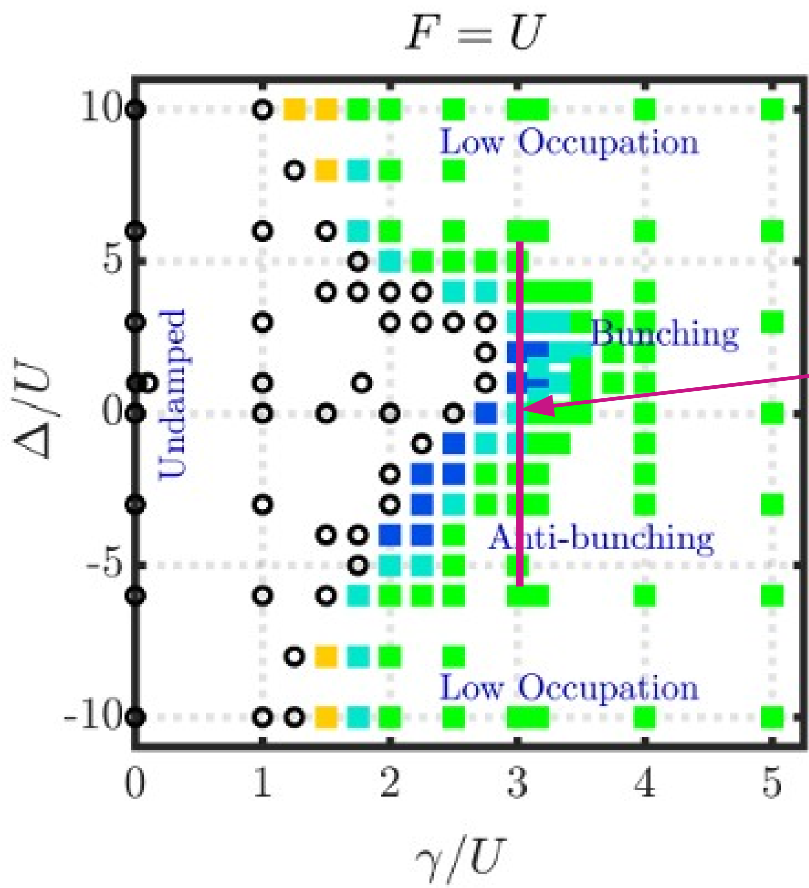
32 site Bose-Hubbard chain
correlation wave after quench



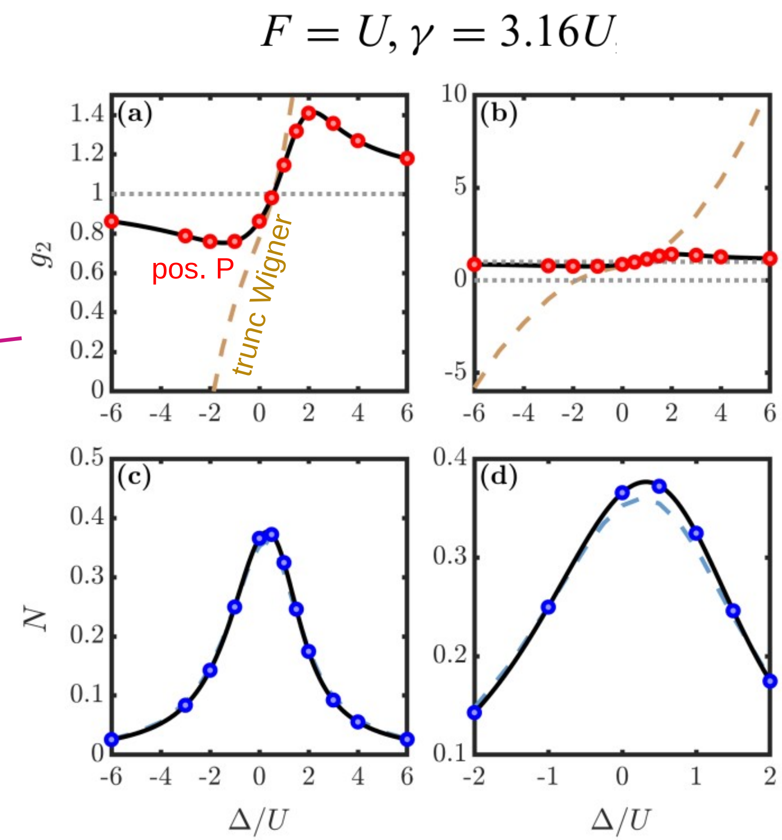
$$g_{1,j}(\tau) = \frac{\langle \widehat{a}_1^\dagger(t_0)\widehat{a}_j^\dagger(t_0+\tau)\widehat{a}_j(t_0+\tau)\widehat{a}_1(t_0)\rangle}{n_1(t_0)n_j(t_0+\tau)}$$

Closer look at occupation dependence

stability



accuracy, comparison to trunc Wigner



$$\gamma \gtrsim 3U\left(\frac{F}{U}\right)^{0.30}$$

$$N \sim \left(\frac{F}{U}\right)^{2/3}$$

resultant occupation dependence of stability region

$$\gamma \gtrsim 3U\sqrt{N}$$