The Wigner Stochastic Gross-Pitaevskii equation

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\[ \hbar \frac{\partial \psi_W(x)}{\partial t} = -i (1 - i \gamma) \left( \mathcal{L}_{GP} - \frac{g}{\Delta v} \right) \psi_W(x) + \sqrt{\gamma \hbar} \left[ 2k_B T + \mathcal{L}_{GP} - \frac{g}{\Delta v} \right] \eta(x, t) + i \sqrt{\gamma \hbar} g \xi(x, t) \psi_W(x) \]

\[ \mathcal{L}_{GP} = -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \mu + g |\psi(x)|^2 \]
The aim

- To describe $T > 0$ degenerate Boson clouds with
  - Small condensate fraction
  - In equilibrium
  - Including *quantum fluctuations*, shot noise

\[
\hat{H} = \int dx \left\{ \hat{\Psi}^\dagger (x) \left[ V(x) - \frac{\hbar^2}{2m} \nabla^2 \right] \hat{\Psi} (x) + \frac{g}{2} \hat{\Psi}^\dagger (x)^2 \hat{\Psi} (x)^2 \right\}
\]
Why and when?

* Low temperature quasicondensates

* Quantum fluctuation effects on solitons, defects

* Better thermal initial states for quantum dynamics

* Unambiguous dependence on N

* Shot noise

* Discretization of N

* Small N
Background: some past approaches

- Bogoliubov
- Classical field
- SGPE
- Truncated Wigner
- SGPE + positive-P
Bogoliubov

Assume dominant condensate

\[ \hat{\Psi}(\mathbf{x}, t) = \phi(\mathbf{x}, t) + \hat{\delta}(\mathbf{x}, t) \]

Neglect high order terms

Effective Hamiltonian:

\[ \hat{H}_{\text{eff}} = \int d^3\mathbf{x} \hat{\delta}^\dagger(\mathbf{x}) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + 2g |\phi(\mathbf{x})|^2 \right] \hat{\delta}(\mathbf{x}) + \frac{g}{2} \int d^3\mathbf{x} \phi(\mathbf{x})^2 \hat{\delta}^\dagger(\mathbf{x})^2 + \text{h.c.} \]

Equation of motion:

Nice and linear

\[ \hbar \frac{\partial \hat{\delta}(\mathbf{x}, t)}{\partial t} = -i \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + 2g |\phi(\mathbf{x}, t)|^2 \right] \hat{\delta}(\mathbf{x}, t) - ig \phi(\mathbf{x}, t)^2 \hat{\delta}^\dagger(\mathbf{x}, t) \]

Fully quantum results, including quantum and thermal fluctuations : )

Only small quantum depletion, very low temperatures : (}

Inaccurate at long times because quasiparticles do not interact.
Classical field and related methods

Incoherent Region (thermal reservoir) $\mu, T$

c-field Region (PGPE) $\mu, T$
\[
\frac{\partial \hat{\rho}_C}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}_C, \hat{\rho}_C \right]
+ \int dx \overline{G}(x) \left\{ [\hat{\phi}^\dagger(x)\hat{\rho}_C, \hat{\phi}(x)] + \left[ 1 - \frac{\mu}{k_BT} \right] \hat{\phi}(x)\hat{\rho}_C + \frac{\hbar}{k_BT} [\hat{L}_C \hat{\phi}(x)]\hat{\rho}_C, \hat{\phi}^\dagger(x) \right\} + \text{h.c.}
\]

The low-energy field \( \hat{\phi} = \mathcal{P}_C \hat{\Psi} \) is the projection of the full Bose field \( \hat{\Psi} \) onto the low energy subspace below the “cutoff” – with projector \( \mathcal{P}_C \).

\[
\hat{H}_C = \int dx \hat{\phi}^\dagger(x) \left[ H_{sp}(x) + \frac{g}{2} \hat{\phi}^\dagger(x)\hat{\phi}(x) \right] \hat{\phi}(x).
\]

including the single-particle Hamiltonian density

\[
H_{sp}(x) = -\frac{\hbar^2}{2m} \nabla^2 + V(x)
\]

and GPE frequency:

\[
\hat{L}_C \hat{\phi}(x) = \frac{1}{\hbar} \mathcal{P}_C \left[ H_{sp}(x)\hat{\phi}(x) + g \hat{\phi}^\dagger(x)\hat{\phi}(x)\hat{\phi}(x) \right]
\]

Assumptions include:

\[
\exp \left[ \frac{\hat{L}_C - \mu}{k_BT} \right] \approx 1 + \frac{\hat{L}_C - \mu}{k_BT}
\]
Classical field

Master equation: ignore coupling to bath, **treat field as classical**

Field evolution equation:

\[ \mathcal{L}_{GP} = -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \mu + g |\psi(x)|^2 \]

\[ \hbar \frac{\partial \psi(x)}{\partial t} = -i \mathcal{L}_{GP} \psi(x) \]

Initial state:

\[ \psi(x, 0) = \phi_0(x) \]

Observables:

\[ \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(x') \rangle = \langle \psi(x)^* \psi(x') \rangle_{\text{ensemble}} \]

Works remarkably well

But, no quantum fluctuations

Temperature found after simulation – system thermalises with itself
SGPE – Stochastic Gross-Pitaevski equation

Master equation: *include coupling to bath, treat field as classical*

Field evolution equation:

\[ \mathcal{L}_{GP} = -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \mu + g |\psi(x)|^2 \]

\[ \hbar \frac{\partial \psi(x)}{\partial t} = -i(1 - i\gamma)\mathcal{L}_{GP}\psi(x) + \sqrt{2\gamma\hbar k_B T}\eta(x, t) \]

Initial state:

\[ \psi(x, 0) = 0 \]

Observables:

\[ \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(x') \rangle = \langle \psi(x)^\ast \psi(x') \rangle_{\text{ensemble}} \]

Same results as classical field in GCE

But, no quantum fluctuations

Temperature set externally – system thermalises with bath. Grand canonical ensemble only
Truncated Wigner

Master equation: ignore coupling to bath, treat field as quantum

Field evolution equation:

\[ \hbar \frac{\partial \psi_W(x)}{\partial t} = -i \mathcal{L}_{GP} \psi_W(x) \]

\[ \mathcal{L}_{GP} = -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \mu + g |\psi(x)|^2 \]

GPE equation

Initial state:

\[ \psi_W(x, 0) = \phi_0(x) + \frac{1}{\sqrt{2}} \chi(x) \]

\[ \langle \chi(x)^* \chi(x') \rangle = \delta(x - x') \]

Observables:

\[ \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(x') \rangle = \langle \psi_W(x)^* \psi_W(x') \rangle_{\text{ensemble}} - \frac{1}{2} \delta(x - x') \]

Quantum fluctuations at short times:)

But, later, don't know what, and how soon. Possibly bogus results: ( 

Late times: classical field ensemble again, quantum fluctuations converted to heat: ( 

Positive-P treatment of SGPE

Swisłocki, Deuar, arXiv:1409.0146

Master equation: include coupling to bath, treat field as quantum

Field evolution equations:

\( L_{PP} = -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \mu + g \tilde{\psi}(x)^* \psi(x) \)

\[ \hbar \frac{\partial \psi(x)}{\partial t} = -i(1 - i\gamma)L_{PP}\psi(x) + \sqrt{i\hbar g(1 - 2i\gamma)} \xi(x, t)\psi(x) + \sqrt{2\gamma\hbar k_B T} \eta(x, t) \]

\[ \hbar \frac{\partial \tilde{\psi}(x)}{\partial t} = -i(1 - i\gamma)L_{PP}^*\tilde{\psi}(x) + \sqrt{i\hbar g(1 - 2i\gamma)} \tilde{\xi}(x, t)\tilde{\psi}(x) + \sqrt{2\gamma\hbar k_B T} \eta(x, t) \]

\[ \langle \eta(x, t)^* \eta(x', t') \rangle = \delta(x - x') \delta(t - t') \]

\[ \langle \xi(x, t)\xi(x', t') \rangle = \delta(x - x') \delta(t - t') \]

Initial state:

\( \psi(x, 0) = \tilde{\psi}(x, 0) = \phi_0(x) \)

Observables:

\( \langle \hat{\Psi}^\dagger(x)\hat{\Psi}(x') \rangle = \langle \text{Re} [\tilde{\psi}(x)^* \psi(x')] \rangle_{\text{ensemble}} \)

Full quantum treatment including fluctuations : )

Only short times because of noise amplification : ( Partly alleviated with SGPE initial conditions, but bogus quench.
The trick would be to

- Have Wigner representation, but without thermalization into a classical field ensemble

OR

- Have positive-P description, but without noise catastrophe

OR

- "Bogoliubov" but without assumption of dominant condensate
Usual route to obtain the SGPE

* Master equation
* Apply Wigner representation
* Obtain FPE for Wigner field
* Truncate high order terms
* Ignore noncommuting pieces
Wigner SGPE (1)

Master equation: include coupling to bath, treat field as quantum (Wigner representation)

Field evolution equations:

\[ \mathcal{L}_{GP} = -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \mu + g |\psi(x)|^2 \]

\[ \hbar \frac{\partial \psi_W(x)}{\partial t} = -i(1-i\gamma) \left[ \mathcal{L}_{GP} - \frac{g}{\Delta v} \right] \psi_W(x) \]

\[ + \sqrt{\gamma \hbar} \left[ 2k_B T + \mathcal{L}_{GP} - \frac{g}{\Delta v} \right] \eta(x,t) + i \sqrt{\gamma \hbar} g \xi(x,t) \psi_W(x) \]

\[ \langle \eta(x,t)^* \eta(x',t') \rangle = \delta(x-x') \delta(t-t') \]

\[ \langle \xi(x,t) \xi(x',t') \rangle = \delta(x-x') \delta(t-t') \]

Turns out to be similar to an equation found in the past by Duine, Stoof, PRA 65, 013603 (2001)

Never tried before. Probably because noise terms look nasty.

However, with some tricks, they become tractable.
Initial state:

\[ \psi_W(x, 0) = 0 + \frac{1}{\sqrt{2}} \chi(x) \]

Observables:

\[ \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(x') \rangle = \langle \psi_W(x)^* \psi_W(x') \rangle_{\text{ensemble}} - \frac{1}{2} \delta(x - x') \]

\[ \langle \hat{\Psi}^\dagger(x) \hat{\Psi}^\dagger(x') \hat{\Psi}(x) \hat{\Psi}(x') \rangle = \langle |\psi_W(x)|^2 |\psi_W(x')|^2 \rangle_{\text{ensemble}} \]

\[ -\frac{1}{2} \left[ \frac{1}{\Delta v} + \delta(x - x') \right] \langle |\psi_W(x)|^2 + |\psi_W(x')|^2 \rangle_{\text{ensemble}} + \frac{1}{4\Delta v} \left[ \frac{1}{\Delta v} + \delta(x - x') \right] \]
Scaling in GPE, SGPE

Field evolution equation:
\[
\mathcal{L}_{GP} = -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \mu + g |\psi(x)|^2
\]

\[
\hbar \frac{\partial \psi(x)}{\partial t} = -i(1 - i\gamma) \mathcal{L}_{GP} \psi(x) + \sqrt{2\gamma \hbar k_B T} \eta(x, t)
\]

Equation is unchanged under the transformation:
\[
g \rightarrow g \lambda
\]
\[
\psi(x) \rightarrow \psi(x) / \sqrt{\lambda}
\]
\[
T \rightarrow T / \lambda
\]

BUT quantum granularity rises with \( \gamma_{LL} \) or \( \lambda \)

Uniform gas dimensionless parameters:
- Interaction strength \( \gamma_{LL} = g/n \sim \lambda^2 \)
- Relative temperature \( \tau = T/4\pi T_d \sim \lambda \)
Test case: 1D trapped Bose gas

- Baseline case (h.o. units):
  \[ \mu = 22.4, \ T = 139 = 0.16T_\Phi, \ g=0.01 \]

- Centrally:
  \[ \gamma_{LL} = 4.5 \times 10^{-6}, \ \tau = 5.5 \times 10^{-5}, \ \tau / \sqrt{\gamma} = 0.026 \]
  [cold quasicondensate, no solitons]

- Then, we change \( \lambda, \ i.e. \ g, \) and \( N \sim 1/g, \ T \sim 1/g, \)
  to keep equivalent SGPE equation, \( \mu \) and \( T/T_\Phi \) constant.

\[
\hbar \frac{\partial \psi(x)}{\partial t} = -i(1 - i\gamma) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \mu + g|\psi(x)|^2 \right] \psi(x) + \sqrt{2\gamma \hbar k_B T} \eta(x, t)
\]
Appearance of vacuum phase fluctuations

As interaction rises, keeping constant chemical potential $\mu$ and $T/T_d$, ....

Small $\gamma_{LL} = g/n$

Larger $\gamma_{LL} = g/n \sim 0.001$

Largest $\gamma_{LL} = g/n \sim 0.01$

**Phase coherence length**


Quantum fluctuations:

$$g_1(r) \approx \rho \left( \frac{r_1}{r} \right)^{1/2} \pi \rho \xi$$

Thermal fluctuations:

$$\ln \left[ \frac{g_1(r) / \rho}{\rho} \right] = \frac{r}{l_c} + K + o \left( 1/r^n \right)$$

The expected drop in phase correlation length

**Classical fields**

**Quantum**

**Thermal**
Antibunching appears

Swisłocki, Proukakis, PD, in preparation

Classical field (SGPE)

$g^{(2)}(0,x)$

bunching

antibunching

Increasing $\gamma_{LL} = g/n$

WSGPE
Strong soliton regime $\tau / \sqrt{\gamma_{LL}} \approx 1.0$, $\mu = 22.4$

SGPE $\gamma_{LL} \to 0$, $N \to \infty$, e.g. $N \approx 20\,000$
Strong soliton regime $\frac{\tau}{\sqrt{\gamma_{LL}}} \approx 1.0, \mu = 22.4$

WSGPE $\gamma_{LL} \approx 0.001 \quad N \approx 2000$
Strong soliton regime $\tau / \sqrt{\gamma_{LL}} \approx 1.0$, $\mu = 22.4$

WSGPE $\gamma_{LL} \approx 0.01 \ N \approx 500$
Cutoff dependence

bunching

$g(2)(0,0)$

- classical field
- thermal fluctuations
- WSGPE
- exact quantum value

$k_{cut}$
• Quantum and thermal fluctuations together
  \textit{Can see quantum depletion, antibunching, ...}

• The simulation is \textit{stable at long times}
  \textit{Finally! : )}

• Behaviour depends explicitly on $N$
  \textit{Starts to be pathological if $N$ is too small, though.}

• Cutoffs still make a difference
  \textit{Also a strange, poorly understood, noise term}

• Can still see sensible single trajectories