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Physica A 266 (1999) 291–298

PHYSICA A

Optimal paths and growth processes

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Abstract

Interfaces in systems with strong quenched disorder are fractal and are thus in a different universality class than the self-affine interfaces found in systems with weak quenched disorder. The geometrical properties of strands arising in loopless invasion percolation clusters, in loopless Eden growth clusters, and in the ballistic growth process are studied and their universality classes are identified.

There has been considerable recent interest in the problem of deducing the best path that connects two sides of a physical region [1–11]. The earliest example was that of a domain wall that divides a two-dimensional Ising ferromagnet at zero temperature into up and down spin domains when antiperiodic boundary conditions are applied in one direction. If the exchange couplings J_{ij} in the Hamiltonian $H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j$, with $S_i = \pm 1$, are ferromagnetic but weakly disordered then the domain wall forms a path which is self-affine. This means that its length, I, scales as

$$I \sim L^1 \tag{1}$$

and its characteristic width, W, as measured by the average transverse end-to-end separation, scales as

$$W \sim L^{\alpha_{DP}} , \tag{2}$$

where L is the characteristic linear size of the system. The roughness exponent α_{DP} corresponds to the so-called directed polymer (DP) universality class [4,5] and it is equal to $\frac{2}{3}$. The value of α_{DP} was deduced initially from numerical calculations [1,2]. Huse et al. [3] demonstrated that this problem, in the continuum limit, maps into both

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0378-4371/99/$ – see front matter
PII: S0378-4371(98)00605-0

Burgers equation [3,6] for non-linear diffusion and the Kardar–Parisi–Zhang (KPZ) equation [7,5,8],

$$\partial h/\partial t = \sigma \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t) \tag{3}$$

for non-linear growth. The KPZ equation describes an interfacial growth process – the interface height h is obtained as a function of the transverse position, x and time t in the presence of an annealed noise η. Here, σ characterizes a surface tension, or a domain wall stiffness, and λ is the non-linearity control parameter influencing lateral growth.

The interfaces generated by the KPZ equation are characterized by three exponents: the growth exponent $\beta = \frac{1}{3}$, the roughness exponent $\alpha = \frac{1}{2}$ corresponding to a saturated stage of the growth, and the dynamical exponent $z_{KPZ} = \alpha/\beta = \frac{3}{2}$ which gives the scaling of the time to reach saturation with L. Overall, the scaling form for W in the KPZ growth is given by $W = L^{\alpha} f(t/L^{z_{KPZ}})$, where f is a scaling function which is a power law for small arguments. The mapping of the DP problem to the KPZ equation yields the result that

$$\alpha_{DP} = 1/z_{KPZ} \, . \tag{4}$$

Examples of discrete processes which are considered to be in the KPZ universality class are the Eden growth and ballistic growth [8]. Here, we consider specific variants of these two growth processes in which the emerging interface is seen as consisting of tips of loopless trees as illustrated in Figs. 1 and 2, respectively. For the Eden growth, we start either with a central seed site (spherical geometry) or line of seed sites (planar geometry) and select randomly one bond that emanates from the starting structure. The selection process is repeated with respect to the new structure and a new bond is selected in a way that disallows formation of loops, i.e. a given site can be accessed by only one bond. Thus the resulting structure is a spanning tree that is the union of strands from the starting set of sites to each of the other sites on the lattice. Note that, when two strands intersect, they overlap the rest of the way to an origin. For ballistic growth in 2D, the starting geometry is planar. One visualizes a 'rain' of incoming particles that come from the top and attach to the first encountered perimeter site.

In Ref. [9], we have provided numerical evidence that the relationship between the Eden growth and the KPZ equation extends beyond the properties of the outer interface: the strands that form the Eden growth cluster are in the DP universality class. Related analytical arguments have been presented by Roux et al. [10]. Thus the Eden strands are self-affine and the roughness exponent is the same as for the DP in a quenched random environment. The effective emergence of the quenched environment is due to the blocking of the new possible growth sites by other growing trees.

Here we demonstrate that a similar blocking is also operational in the case of 2D ballistic growth and it also leads to the self-affine strands with a roughness exponent which is consistent with $\frac{2}{3}$. This suggests that probably all growth processes with interfaces belonging to the KPZ universality class can be considered as unions of

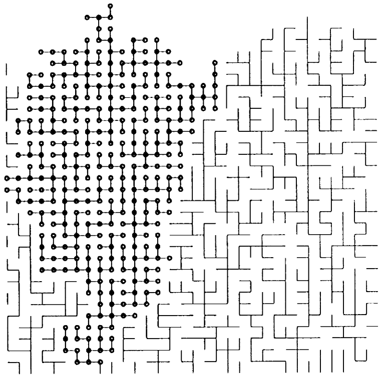

Fig. 1. Eden growth process which starts from a flat initial state consisting of a line spannning 32 sites. The growth consists of several trees with roots in the first layer (bottom line). One of these trees is enhanced graphically by using darker symbols.

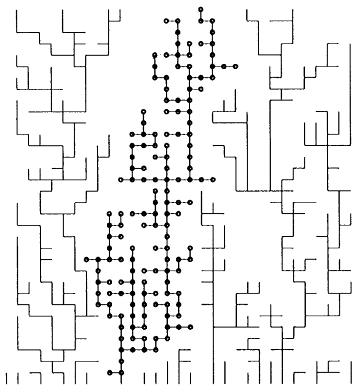

Fig. 2. Same as in Fig. 1 but for a ballistic growth process.

strands whose geometry agrees with that of the optimal paths in systems with a weak quenched disorder. Fig. 3 shows that the scaling of W for the strands in ballistic growth is identical to that found for Eden growth, independent of whether the Eden clusters are grown in a spherical or planar fashion.

Fig. 4, shown in Ref. [9], summarizes results on the D-dependence of the roughness exponent for the Eden growth strands. The data obtained for D=2,3 and 4 suggest that α_{DP} may approach $\frac{1}{2}$ in D=6.

It should be noted that the concepts of domain walls and optimal paths diverge for $D > 2$. Middleton [11] has used an exact minimum-cut algorithm to demonstrate that the domain wall in weakly disordered ferromagnets in 3 and 4D are self-affine and the corresponding roughness exponents are 0.41 ± 0.01 and 0.22 ± 0.01.

Can there be any other universality classes of optimal paths and optimal domain walls? Consider volume-dependent exchange couplings

$$J_{ij} = yc^{(y-1/2)L^D}\,, \tag{5}$$

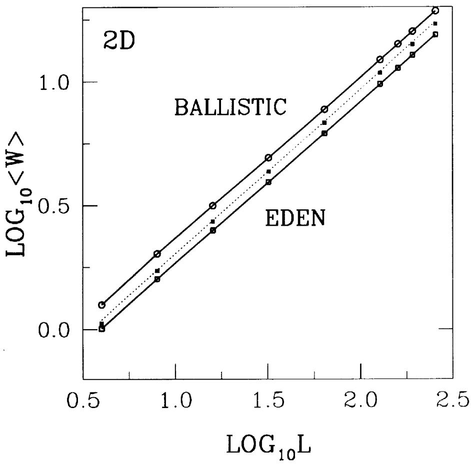

Fig. 3. The transverse end-to-root distance of each invaded site versus its vertical distance from the base for Eden and ballistic growth processes on the square lattice generated from a base of 256 sites. The data points are averaged over 10 000 realizations. The slopes indicated by solid lines correspond to α=0.65. The dotted line, with the slope of 0.66, corresponds to the Eden growth obtained in Ref. [9] from a central seed site, again on the square lattice. In this case, the data points indicate the transverse end-to-center distance of the most forward paths. The data points are averaged over 42 000 realizations for L between 4 and 128 and over 12 000 realizations for L=192 and 256.

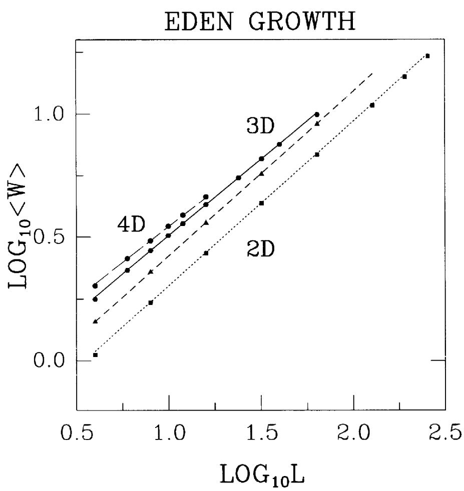

Fig. 4. The transverse end-to-center distance of the most forward paths, generated in the Eden growth in a spherical geometry on D-dimensional lattice as a function of the longitudinal distance, L. The figure represents a summary of the results obtained in Ref. [9]. The square symbols connected by the dotted line are the same as those in Fig. 3. The triangle, hexagon, and circular symbols correspond to the triangular, cubic, and 4 D hypercubic lattices, respectively. The slopes indicated correspond to 0.66 ± 0.01, 0.62 ± 0.02, and 0.59 ± 0.02 for the 2, 3, and 4 D lattices respectively. The statistics are listed in Ref. [9]. The average length of the strands scales linearly with L.

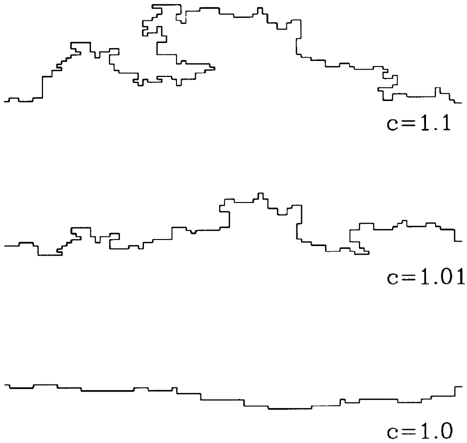

Fig. 5. Domain walls in a two-dimensional random Ising ferromagnet (L=96) with the exchange couplings given by Eq. (5) for the three values of c indicated. The figure comes from Ref. [12].

where y is a random number from the interval $]0,1]$ and $c \geqslant 1$. The values of J_{ij} vary between $J_{\min} = y_{\min} c^{-L^D/2}$ and $J_{\max} = c^{L^D/2}$. The corresponding probability distribution for the couplings is $P(J) = [J \ln(J_{\max}/J_{\min})]^{-1}$. We [12] have investigated properties of domain walls in $2D$ ferromagnets endowed with such couplings using the minimum-cut algorithm. As illustrated in Fig. 5, the interface is self-affine, and in the DP universality class, when viewed on length scales which are smaller than a crossover length, l_c. However, on distances exceeding l_c, the interfaces become fractal:

$$W \sim L^1 , \qquad I \sim L^{D_f} \tag{6}$$

with D_f consistent with the value of 1.22 ± 0.01. The length l_c diverges to ∞ when $c \to 1$ but it becomes small already for c of order 1.1. Thus, in the limit of large c, the domain wall is fractal on all length scales.

This new universality class of strong disorder, introduced in Ref. [13], arises when the exchange couplings are so widely distributed that the total energetic cost of a path is equal to the largest bond belonging to the path. In this limit, the cost space has an ultrametric structure: the minimum cost for connecting site A with site B, $C(A,B)$ satisfies the relation

$$C(A,B) \leqslant MAX(\ C(A,X), C(X,B)\ ) \tag{7}$$

for an arbitrary site X. The property of ultrametricity allows for an exact determination of the non-trivial ground state both for ferromagnetic couplings with antiperiodic boundary conditions and for spin glasses, when the sign of the couplings is chosen randomly to be either positive or negative. The ground state construction proceeds by rank ordering the bonds and then by building connectivity clusters by deciding the state of the most strongly connected spins first and then by allowing for cluster flips when initially independent clusters merge. This construction becomes exact in the limit in which, whenever a new bond is considered, its effect is significantly weaker than

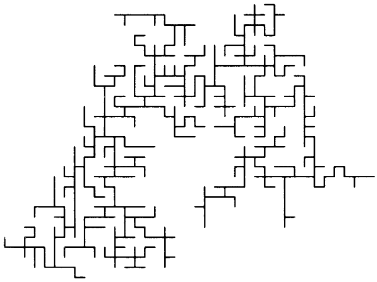

Fig. 6. An example of a loopless IP growth process from a single central site on the 64×64 lattice. Here, the growth is stopped once the boundary on the right is reached.

that of the bonds considered before. This property allowed Newman and Stein [14] to prove theorems about the D-dependence of the ground state degeneracies in model spin glass systems.

The spin configuration for spin glasses and ferromagnets are necessarily different, for a given type of boundary conditions. However, the domain walls generated by imposing antiperiodic boundary conditions in one direction are identical for the two systems if the placement of the couplings according to their strength is the same. We have determined the fractal dimensionality of the domain walls in the strong disorder limit using the procedure based on rank ordering and determined D_f to be 1.22 ± 0.02 and 2.5 ± 0.05 for 2 and 3 D walls, respectively.

These domain walls define a novel kind of percolation (on the dual lattice). Since a domain wall breaks bonds which are possibly the weakest but consistent with the condition that side-to-side connectivity is established we can determine its shape in one of three following ways. (1) Consider the lattice with no bonds present, rank order the exchange couplings and keep inserting the bonds starting from the weakest end and stopping once the side-to-side connectivity is established. At this point we prune off all the bonds, in reverse order, that do not break the connectivity. (2) Start with all the bonds present, keep removing them to the point that one more removal would break the connectivity, remove all remaining bonds that do not break the connectivity. (3) Grow an invasion percolation (IP) cluster [8,15] from one side until the second side is reached and then remove all bonds which are spurious for connectivity. Each of these prescription generates the same geometrical object. This object is a topologically one-dimensional path if the side-to-side connectivity is obtained just through the bonds. In order to obtain a topologically $2D$ domain wall in the three-dimensional space, there should be a sufficient number of bonds to provide connectivity through plaquettes.

The variant of the IP growth process that we consider here generates loopless clusters without trapped regions, as illustrated in Fig. 6. The process is defined as follows. We preassign quenched random numbers to the bonds (like the specific values of the J_{ij}'s) and define a starting region (a site or a set of sites). We consider all possible bonds that the invasion can take place into and pick the bond with

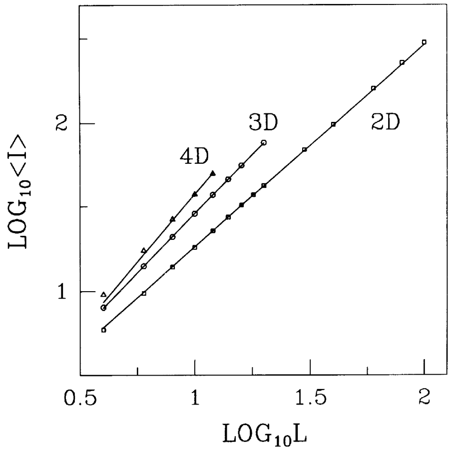

Fig. 7. Average length of the strands in the IP clusters grown bond-by-bond from a central site versus the spanning distance L. Results for a similarly defined site IP problem are almost identical. The slopes are 1.22 ± 0.01, 1.42 ± 0.02, and 1.59 ± 0.02 for D equal to 2, 3, and 4 respectively. The statistics are as listed in Ref. [9].

the smallest number assigned to it. The invasion proceeds only to previously uninvaded sites.

In $2D$, the domain wall of a strongly disordered system is a particular strand of an IP cluster. For all dimensionalities, any IP cluster is a union of strands with similar but D-dependent geometrical properties. We have carried out detailed numerical studies of such strands for D=2, 3, and 4 [9]. The results, as illustrated in Fig. 7, yield D_f close to $(D+4)/5$. In $2D$, $D_f = 1.22 \pm 0.01$, which is identical (the error bars are smaller) to the result obtained based on studying the optimal paths. These values of D_f are universal – they do not depend on the type of the lattice and on whether the invasion is defined in terms of bonds, as described here, or sites.

Above D=6, the optimal paths in the strong disorder limit is expected to acquire the geometry of an uncorrelated random walk which is consistent with the upper dimensionality of 6 for the IP problem (see, e.g., Ref. [16]). Since the IP leads to fractal paths due to the quenched character of the randomness, one might expect that the transverse wandering of its strand must be greater than that for the Eden strand of the same length, suggesting that $1/D_f \geqslant \alpha_{DP}$. At the same time, $\alpha_{DP} \geqslant \frac{1}{2}$ since the transverse wandering of the DP in a quenched random medium ought to be larger than that of a random walk with the same number of steps. Accepting the conjecture that the Eden strand is in the same universality class as a DP in a weakly disordered quenched environment, we deduce that the upper critical dimensionality of the DP problem, and hence the KPZ equation, should be equal to 6 (i.e. 5+1) because for that D the fractal dimensionality D_f becomes 2.

It should be noted that the geometry of the optimal path in the strong disorder limit is distinct from that of the shortest geometrical path on a percolating backbone [17].

The latter is calculated by eliminating longer legs of any loop in a percolation backbone, with disregard to an energy cost. However, Porto et al. [18] have shown recently that if one considers an IP with trapping (which corresponds to the defending medium being incompressible, unlike the case considered here) then the shortest path is in the same universality class as the optimal path. Furthermore, the IP with and without trapping may be in different universality classes for $D=2$ and 3.

It is interesting to mention the recent result of Barabasi [19] that an IP way of generating a loopless tree (or a lattice animal) that spans a predetermined set of sites provides a tree which is energetically optimal. The paths themselves, however, become optimal only in the strong disorder limit.

The strongly disordered systems have been invoked in the literature in the contexts of transport [20] and magnetic properties [21,22] of solids at low temperatures, and that of electrical and hydraulic conduction in porous media [23].

This work was supported by KBN (Grant No. 2P03B-025-13), NASA, and the Donors of the Petroleum Research Fund administered by the American Chemical Society.

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