Topological Insulators and the Quantum Spin Hall Effect

Charles L. Kane, University of Pennsylvania

I. Introduction
   - Topological band theory and topological insulators

II. Two Dimensions : Quantum Spin Hall Effect
   - Time reversal symmetry and edge states
   - Experiment: Transport in HgTe quantum Wells

III. Three Dimensions : Topological Insulators
   - Topological Insulator Surface States
   - Experiment: Photoemission on Bi$_x$Sb$_{1-x}$, Bi$_2$Se$_3$, Bi$_2$Te$_3$

IV. Superconducting proximity effect, Majorana fermions
   - Majorana fermions
   - A route to topological quantum computing?

Thanks to Gene Mele, Liang Fu, Jeffrey Teo
The Insulating State

The Integer Quantum Hall State

IQHE with zero net magnetic field

Graphene with a periodic magnetic field \( B(\mathbf{r}) \)

\[ B(r) = 0 \]

Zero gap, Dirac point

\[ B(r) \neq 0 \]

Energy gap

\[ \sigma_{xy} = e^2/h \]

\[ E_g \sim 1 \text{ eV} \]

\[ E_g = \hbar \omega_c \]

\[ \sigma_{xy} = e^2/h \]
Topological Band Theory

The distinction between a conventional insulator and the quantum Hall state is a topological property of the manifold of occupied states

\[ |\Psi(\vec{k})\rangle : \text{Brillouin zone } (T^2) \rightarrow \text{Hilbert space} \]

Classified by the TKNN (or Chern) topological invariant (Thouless et al, 1984)

\[ n = \frac{1}{2\pi i} \int_{BZ} d^2k \cdot \left( \nabla_k u(k) \right) \times \left( \nabla_k u(k) \right) \]

Insulator : \( n = 0 \)
IQHE state : \( \sigma_{xy} = n e^2/h \)

The TKNN invariant can only change when the energy gap goes to zero

Edge States at a domain wall

Gapless Chiral Fermions
Topological Insulator: A New B=0 Phase

2D Time reversal invariant band structures have a $\mathbb{Z}_2$ topological invariant, $\nu = 0,1$

$\nu=0$: Conventional Insulator

$\nu=1$: Topological Insulator

$$E_{k^*=0}, E_{k^*={\pi}/a}$$

Edge States

Kramers degenerate at time reversal invariant momenta

$$k^* = -k^* + G$$

$\nu$ is a property of bulk bandstructure. Easiest to compute if there is extra symmetry:

1. $S_z$ conserved: independent spin Chern integers: $n_\uparrow = - n_\downarrow$ (due to time reversal)

Quantum spin Hall Effect:

$$\nu = n_{\uparrow,\downarrow} \mod 2$$

2. Inversion (P) Symmetry: determined by Parity of occupied 2D Bloch states at $\Gamma_{1,2,3,4}$

$P|\psi_n(\Gamma_i)\rangle = \xi_n(\Gamma_i)|\psi_n(\Gamma_i)\rangle$

$\xi_n(\Gamma_i) = \pm 1$

$$(-1)^\nu = \prod_{i=1}^{4} \prod_{n} \xi_{2n}(\Gamma_i)$$
Two dimensions: Quantum Spin Hall Insulator

I. Graphene, Kane, Mele PRL '05

- Intrinsic spin orbit interaction
  \( \Rightarrow \) small (~10mK-1K) band gap
- \( S_z \) conserved: “| Haldane model |^2”
- Edge states: \( G = 2\frac{e^2}{h} \)

II. HgCdTe quantum wells

Theory: Bernevig, Hughes and Zhang, Science '06
Experiment: Konig et al. Science '07

- \( d < 6.3 \text{ nm} \):
  Normal band order
- \( d > 6.3 \text{ nm} \):
  Inverted band order

\( G \approx 2\frac{e^2}{h} \) in QSHI
3D Topological Insulators

There are 4 surface Dirac Points due to Kramers degeneracy

\[ \Lambda_1 \quad \Lambda_2 \quad \Lambda_3 \quad \Lambda_4 \]

Surface Brillouin Zone

2D Dirac Point

\[ E_k = \Lambda_a \quad k=\Lambda_a \]
\[ E_k = \Lambda_b \quad k=\Lambda_b \]

How do the Dirac points connect? Determined by 4 bulk \( Z_2 \) topological invariants \( \nu_0 ; (\nu_1 \nu_2 \nu_3) \)

\( \nu_0 = 1 \) : Strong Topological Insulator

Fermi circle encloses odd number of Dirac points

Topological Metal

\( \nu_0 = 0 \) : Weak Topological Insulator

Fermi circle encloses even number of Dirac points

Related to layered 2D QSHI

\( (\nu_1 \nu_2 \nu_3) \) : Miller indices for stacking direction

Determine location of surface Dirac Points
**Bi$_{1-x}$Sb$_x$**

**Theory:** Predict Bi$_{1-x}$Sb$_x$ is a topological insulator by exploiting inversion symmetry of pure Bi, Sb (Fu, Kane PRL'07)

**Experiment:** ARPES (Hsieh et al. Nature ’08)

- Bi$_{1-x}$Sb$_x$ is a Strong Topological Insulator $\nu_0; (\nu_1, \nu_2, \nu_3) = 1 ; (111)$
- 5 surface state bands cross $E_F$ between $\Gamma$ and $M$

**Bi$_2$Se$_3$**

**ARPES Experiment:** Y. Xia et al., Nature Phys. (2009).


- $\nu_0; (\nu_1, \nu_2, \nu_3) = 1 ; (000)$ : Band inversion at $\Gamma$
- Energy gap: $\Delta \sim .3$ eV : A room temperature topological insulator
- Simple surface state structure:
  Similar to graphene, except only a single Dirac point
Unique Properties of Surface States

Spin polarized Fermi circle (points)
- “Half” an ordinary 2DEG (1DEG)
- Spin and velocity correlated
  - Charge Current ~ Spin Density
  - Spin Current ~ Charge Density

1D Edge States of QSHI
- Elastic backscattering forbidden
- No 1D Anderson localization
- Weak interactions irrelevant

2D Surface States of STI
- $\pi$ Berry’s phase
- Weak antilocalization
- Impossible to localize (Klein paradox)

Exotic States when broken symmetry leads to surface energy gap:
Proximity Effects: Open energy gap at surface

Dirac Surface States: Protected by Symmetry

\[ H_0 = \psi^\dagger (-i v \vec{\sigma} \cdot \vec{\nabla} - \mu) \psi \]

1. Magnetic: (Broken Time Reversal Symmetry)
   - Orbital Magnetic field:
     \[ p \rightarrow p - eA \]
   - Zeeman magnetic field:
     \[ M \psi^\dagger \sigma_z \psi \]
   - Half Integer quantized Hall effect:
     \[ \sigma_{xy} = (n + \frac{1}{2}) \frac{e^2}{h} \]

2. Superconducting: (Broken U(1) Gauge Symmetry)
   - S-wave superconductor
   - Resembles spinless p+ip superconductor
   - Supports Majorana fermion excitations

Fu, Kane PRL 07
Qi, Hughes, Zhang PRB (08)

Fu, Kane PRL 08
Majorana Fermion

• Particle = Antiparticle : $\gamma = \gamma^\dagger$

• Real part of Dirac fermion : $\gamma = \Psi + \Psi^\dagger$; $\Psi = \gamma_1 + i \gamma_2$ “half” an ordinary fermion

• Mod 2 number conservation / $Z_2$ Gauge symmetry : $\gamma \Rightarrow \pm \gamma$

Potential Hosts :

Particle Physics :

• Neutrino (maybe) Allows neutrinoless double $\beta$-decay

Condensed matter physics : Possible due to pair condensation $\langle \Psi^\dagger \Psi^\dagger \rangle \neq 0$

• Quasiparticles in fractional Quantum Hall effect at $\nu=5/2$
• $h/4e$ vortices in p-wave superconductor $Sr_2RuO_4$
• $s$-wave superconductor / Topological Insulator

Current Status : NOT OBSERVED
Majorana Fermions and Topological Quantum Computation

• 2 separated Majoranas = 1 fermion: \( \Psi = \gamma_1 + i \gamma_2 \)
  
  2 degenerate states (full or empty)
  
  1 qubit

• 2N separated Majoranas = N qubits

• Quantum information stored non locally
  
  Immune to local sources decoherence

• Adiabatic “braiding” performs unitary operations

\[ |\psi_a\rangle \rightarrow U_{ab} |\psi_b\rangle \]
Majorana Bound States on Topological Insulators

1. $\hbar/2e$ vortex in 2D superconducting state

Quasiparticle Bound state at $E=0$

Majorana Fermion $\gamma_0$ “Half a State”

2. Superconductor-magnet interface at edge of 2D QSHI

$m = |\Delta_S| - |\Delta_M|$

$E_{\text{gap}} = 2|m|$
1D Majorana Fermions on Topological Insulators

1. 1D Chiral Majorana mode at superconductor-magnet interface

\[ \gamma_k = \gamma_{-k}^\dagger : \text{“Half” a 1D chiral Dirac fermion} \]

2. S-TI-S Josephson Junction

\[ H = -i\hbar v_F \gamma \partial_x \gamma \]

Gapless non-chiral Majorana fermion for phase difference \( \phi = \pi \)

\[ H = -i\hbar v_F \left( \gamma_L \partial_x \gamma_L - \gamma_R \partial_x \gamma_R \right) + i\Delta \cos(\phi/2) \gamma_L \gamma_R \]
Manipulation of Majorana Fermions

Control phases of S-TI-S Junctions

Tri-Junction:
A storage register for Majoranas

Create
A pair of Majorana bound states can be created from the vacuum in a well defined state $|0\rangle$.

Braid
A single Majorana can be moved between junctions. Allows braiding of multiple Majoranas

Measure
Fuse a pair of Majoranas. States $|0,1\rangle$ distinguished by
- presence of quasiparticle.
- supercurrent across line junction
A $Z_2$ Interferometer for Majorana Fermions

A Signature for Neutral Majorana Fermions Probed with Charge Transport

- Chiral electrons on magnetic domain wall split into a pair of chiral Majorana fermions
- "$Z_2$ Aharonov Bohm phase" converts an electron into a hole
- $\frac{dI_D}{dV_s}$ changes sign when $N$ is odd.

$c^\dagger = \gamma_1 - i\gamma_2$
$c = \gamma_1 + i\gamma_2$

Fu and Kane, PRL ‘09
Akhmerov, Nilsson, Beenakker, PRL ‘09
Conclusion

• A new electronic phase of matter has been predicted and observed
  - 2D: Quantum spin Hall insulator in HgCdTe QW’s
  - 3D: Strong topological insulator in Bi$_{1-x}$Sb$_x$, Bi$_2$Se$_3$, Bi$_2$Te$_3$

• Superconductor/Topological Insulator structures host Majorana Fermions
  - A Platform for Topological Quantum Computation

• Experimental Challenges
  - Charge and Spin transport Measurements on topological insulators
  - Superconducting structures:
    - Create, Detect Majorana bound states
  - Magnetic structures:
    - Create chiral edge states, chiral Majorana edge states
    - Majorana interferometer

• Theoretical Challenges
  - Further manifestations of Majorana fermions and non-Abelian states
  - Effects of disorder and interactions on surface states