Classical Graphs
Random Quantum States & Unitary matrices

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in collaboration with

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What is this talk about?

A **non-relativistic quantum theory** of finite systems

We study **quantum correlations** between subsystems.

Consider a **quantum system** S described in a **finite dimensional** complex Hilbert space $\mathcal{H}_d$.

Assume that the system S consists of m subsystems $S_1, S_2, \ldots, S_m$. Then the Hilbert space has the structure of an ***m*–fold tensor product**

$$\mathcal{H}_d \;:=\; \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_m$$

of dimension $d = N_1 \times N_2 \times \cdots \times N_m$.

Assume we do not know the quantum dynamics

What can be said about the quantum state which evolves unitarily in time,

$$|\psi(t)\rangle \;=\; U(t)\,|\psi(0)\rangle \;?$$

Unitary Dynamics & Quantum Chaos

'Quantum chaology': analogues of classically chaotic systems

Quantum analogues of classically chaotic dynamical systems can be described by random matrices

a). autonomous systems – Hamiltonians:

Gaussian ensembles of random Hermitian matrices, (GOE, GUE, GSE)

b). periodic systems – evolution operators:

Dyson circular ensembles of random unitary matrices, (COE, **CUE**, CSE)

Universality classes

Depending on the symmetry properties of the system one uses ensembles form orthogonal ($\beta = 1$); unitary ($\beta = 2$) and symplectic ($\beta = 4$) ensembles.

The exponent β determines the level repulsion, $P(s) \sim s^\beta$ for $s \to 0$ where s stands for the (normalised) level spacing, $s_i = \phi_{i+1} - \phi_i$.

see e.g. F. Haake, *Quantum Signatures of Chaos*

Random Pure states in $\mathcal{H}_N$

'Quantum chaotic' dynamics **(pseudo-random evolution)**

described by a **random unitary** matrix U acting on a pure state produces (almost surely) a **'generic pure state'** $|\psi\rangle = U|\phi_0\rangle$.

• Formally one defines an (unique) **Fubini–Study measure** μ on complex projective spaces which is **unitarily invariant**: for any (measurable) set A of states one requires $\mu(A) = \mu(U(A))$.

• This measure covers the entire space $\mathbb{C}P^{N-1}$ **uniformly**, and for $N = 2$ it is just equivalent to the uniform, **Lebesgue measure** on the Bloch sphere S^2.

How to obtain a **random pure state** $|\psi\rangle$ **?**

a) Take a column (a row) of a **random unitary** U so that $|\psi\rangle = U|i\rangle$.
b) generate N **independent complex random numbers** z_i according to the **normal** distribution. Write $|\psi\rangle = \sum_{i=1}^{N} c_i|i\rangle$ where the expansion coefficients read $c_i = z_i/\sqrt{\sum_i |z_i|^2}$.

One quantum state fixed, one random...

Fix an arbitrary state $|\psi_1\rangle$. Generate randomly the other state $|\psi_2\rangle$.

- What is the average angle χ between these states ?
- What is the distribution $P(\chi)$ of the angle $\chi \ := \ \arccos|\langle\psi_1|\psi_2\rangle|$?

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Measure concentration phenomenon

'Fat hiper-equator' of the sphere S^N in $\mathbb{R}^{N+1}$...
It is a consequence of the Jacobian factor for expressing the volume element of the $N-$ sphere. Let $z = \cos\vartheta_1$, so that

$$J \sim (sin\ \vartheta_1)^{N-1} J_2(\vartheta_2, \ldots, \vartheta_N)$$

Hence the typical angle χ is 'close' to $\pi/2$ and two 'typical random states' are orthogonal and the distribution $P(\chi)$ is **'close' to** $\delta(\chi - \pi/2)$.

How close?

Quantitative description of **Measure Concentration**

Levy's Lemma (on higher dimensional spheres)

Let $f : S^N \to \mathbb{R}$ be a **Lipschitz function**,
with the constant η and the mean value $\langle f \rangle = \int_{S^N} f(x) d\mu(x)$.
Pick a point $x \in S^N$ **at random from the sphere**. For large N it is then **unlikely** to get a value of f much different then the average:

$$P\Big(|f(x) - \langle f \rangle| > \alpha\Big) \;\leq\; 2\exp\Big(-\frac{(N+1)\alpha^2}{9\pi^3\eta^2}\Big)$$

Simple application: the distance from the 'equator'

.

Take $f(x_1, ...x_{N+1}) = x_1$. Then **Levy's Lemma** says that the probability of finding a random point of S^N outside a band along the **equator of width** 2α converges **exponentially** to zero as $2\exp[-C(N+1)\alpha^2]$.

As $N >> 1$ then **every equator** of S^N is **'FAT'**.

Composed systems & entangled states

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states not of the above product form.

Two–qubit system: $d = 2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

Entanglement measures

For any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ define its partial trace $\sigma = \mathrm{Tr}_B |\psi\rangle\langle\psi|$.
Definition: Entanglement entropy of $|\psi\rangle$ is equal to von Neumann entropy of the partial trace

$$E(|\psi\rangle) := \; -\mathrm{Tr}\; \sigma \ln \sigma$$

The more mixed partial trace, the more entangled initial pure state...

Generic pure states of a bi-partite system

'Two quNits' $=$ $N \times N$ quantum system

The space $\mathbb{C}P^{N^2-1}$ of all states in $\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_N$ has $d_{\rm tot} = N^2 - 2$ dimensions.
The subspace of **separable (product) states** $\mathbb{C}P^{N-1} \times \mathbb{C}P^{N-1}$ has only $d_{\rm sep} = 2(N-2)$ dimensions. For large N we observe that $d_{\rm sep} \sim 2N << d_{\rm tot} \sim N^2$ so the **separable states** form a set of measure zero in the space of all states.

Thus a **'typical'** random state is **entangled**!

How much entangled?

Mean entropy of the reduced density matrix ρ

Let us call $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Take any **pure state** $|\psi\rangle \in \mathcal{H}$ and define its partial trace $\rho := {\rm Tr}_B |\psi\rangle\langle\psi| = {\rm Tr}_A |\psi\rangle\langle\psi|$.
The **von Neumann entropy** S of the reduced **mixed state** ρ is a measure of **entanglement** of the initially **pure** bi-partite state $|\psi\rangle$.

Concentration of entropy of the partial trace

Consider an $N \times K$ system with $K \geq N$

The maximal entropy (achieved for $\rho_* = \mathbb{1}_N/N$) is equal to $S_{\max} := \ln N$. Since the **mean entropy**, $\langle S \rangle_\psi \approx S_{\max} - \frac{N}{2K}$, is close to the maximal value a **concentration effect** has to occur...

Levy's lemma and concentration of entanglement

Consider the sphere S^{2NK-1} which represents pure states of a $N \times K$ system with $K \geq N \geq 3$. Use **Levy's lemma** with $f = S(\rho)$. It implies

$$P\Big(S(\mathrm{Tr}_B|\psi\rangle\langle\psi|) < \ln N - N/K - \alpha\Big) \;\leq\; \exp\Big(-\frac{(NK-1)}{8(\pi \ln N)^2}\alpha^2\Big)$$

Hayden, Leung, Winter (2006)

Thus the **reduced density matrix** ρ is close to the maximally mixed state $\rho_* = \mathbb{1}_N/N$, while the initial **random pure state** is close to a **maximally entangled state** $|\psi^+\rangle$ with entropy $S_{\max} = \ln N$.

Graphs & Quantum graphs

How a **quantum graph** looks like ?

Graphs & Quantum graphs

How a **quantum graph** looks like ?

existing **quantum graph**
by **Leszek Sirko**
and his team in Warsaw ....

A different kind of quantum graphs discussed here:

Ensembles of

a) random quantum states
b) random unitary matrices

associated with a given **graph**.

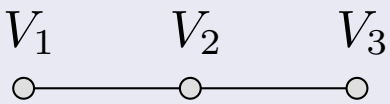
Multi–partite systems:
a) random states associated with a graph

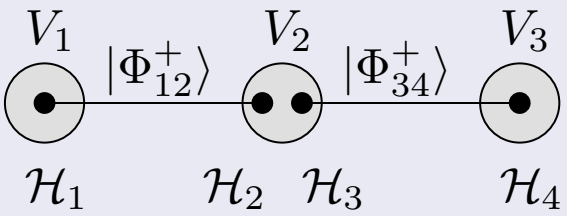
Graph random states

Consider a graph Γ consisting of m edges $B_1, \ldots B_m$ and k vertices $V_1, \ldots V_k$. It represents a composite **quantum system** consisting of $2m$ sub–systems described in the Hilbert space with $2m$–fold tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{2m}$ of dimension N^{2m}.

Each **edge** represents the **maximally entangled state** $|\Phi^+\rangle$ in both subspaces, while each **vertex** represents a **random unitary matrix U** (**Haar measure ='generic' Hamiltonian**), coupling connected systems.

A simple example: three vertices & two edges





We define a **random state** $|\psi\rangle = (\mathbf{U_1} \otimes \mathbf{U_{23}} \otimes \mathbf{U_4})\, |\mathbf{\Phi^+_{12}}\rangle \otimes |\mathbf{\Phi^+_{34}}\rangle$

where $|\Phi^+_{kj}\rangle$ denotes the maximally entangled state in subspaces k, j.

Multi–partite graph systems: mixed states

Partial trace over certain subspaces

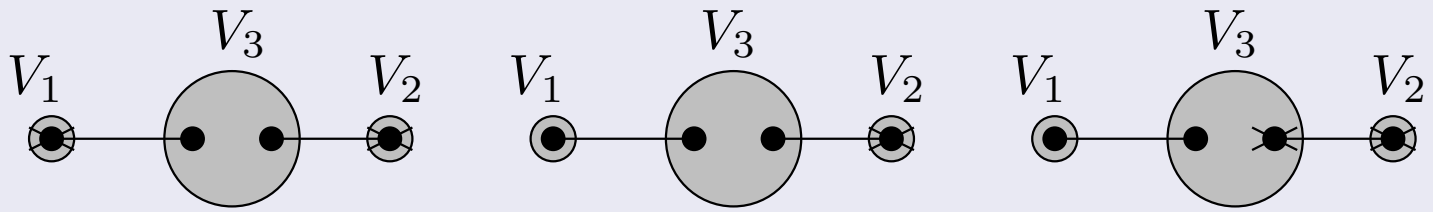
Consider an **ensemble of random pure states** $|\psi\rangle$ corresponding to a given graph Γ. Select a fixed **subset** T of subspaces and define a (random) **mixed state** $\rho(T) = \mathrm{Tr}_T |\psi\rangle\langle\psi|$.

Tasks

• Determine the **spectral properties** of the ensemble of mixed states $\rho(T)$ associated with the graph Γ.

• Find the mean **entropy** $\langle S(\rho)\rangle_\psi$ of the reduced state ρ averaged over the ensemble of graph random pure states $|\psi\rangle_{\Gamma,T}$.

Examples of partial trace for the graph Γ

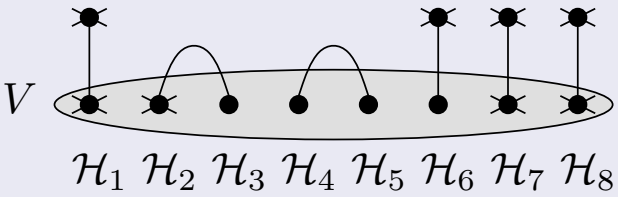


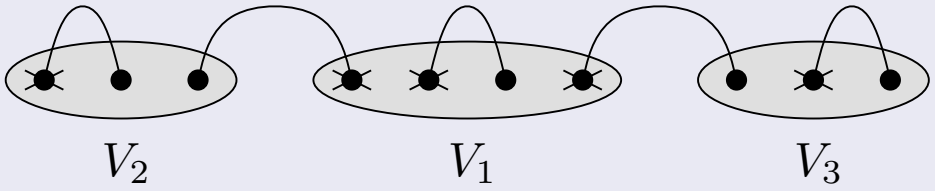
The partial trace is taken over all the subspaces T represented by open symbols.

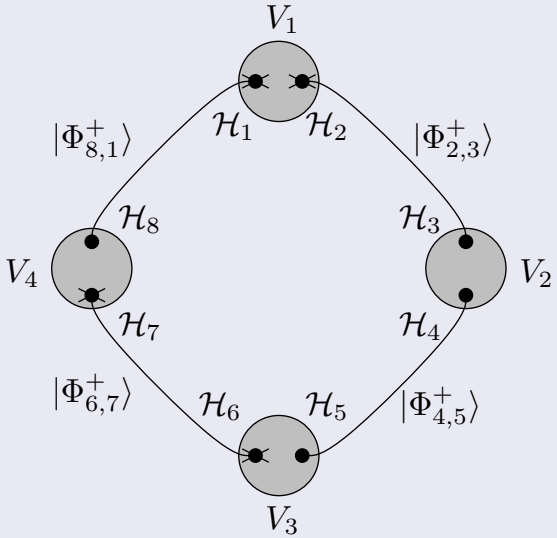
Graphs and random multi–partite systems

Partial trace over certain subspaces

For ensembles of **random states** associated with certain **graphs** Γ and selected subspaces T – cross ($\times$) – over which the partial trace takes place







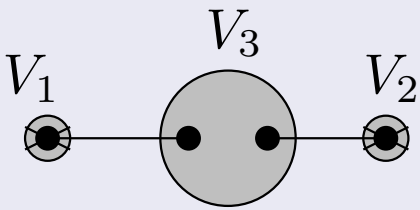
one can compute **moments of the traces** $\mu_q := \langle \mathrm{Tr}\rho^q \rangle_\psi$
and then obtain bounds for the **average entropy** $\langle S \rangle = \langle -\mathrm{Tr}\rho \ln \rho \rangle_\psi$.

Collins, Nechita, K.Ż, *J. Phys.* **A** (2010)

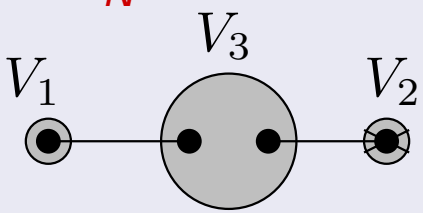
Spectral properties of random mixed states I

Example 1: 2 bonds, 4 subsystems and one bi-partite interaction U_0

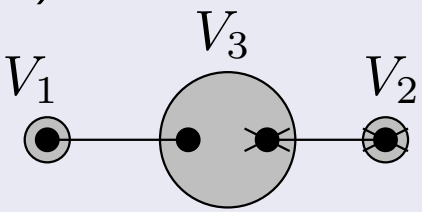
a) h_0 – maximaly mixed state $\rho = \frac{1}{N}\mathbb{1}$ with **entropy** $S(\rho) = \ln N$



or



b) h_1 random mixed state generated according to the induced measure



with **entropy** $S(\rho) \approx \ln N - 1/2$

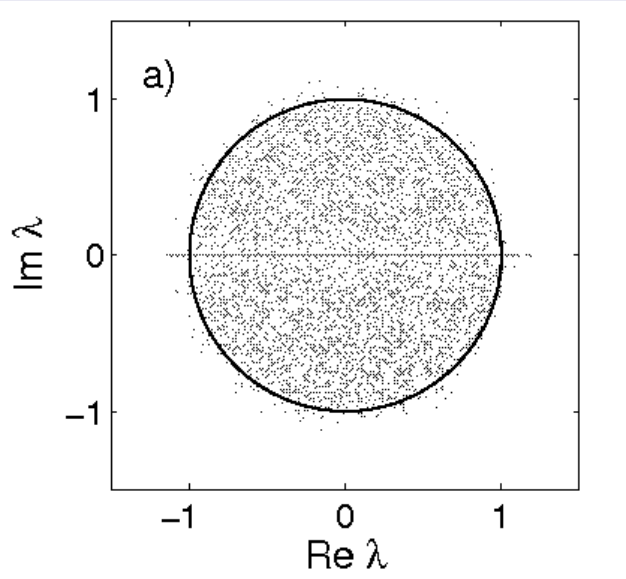
Let $|\psi\rangle = \sum_i \sum_j C_{ij}|i\rangle \otimes |j\rangle$ be a **random pure state.**

Then C is a random matrix of **Ginibre ensemble** (not hermitian!) consisting of independent complex Gaussian entries. The only constraint is its norm: since $\langle\psi|\psi\rangle = 1$ then $|C|^2 = \mathrm{Tr}CC^\dagger = 1$.

What is the distribution of eigenvalues of a **non–hermitian matrix** C and of a positive **Wishart** matrix $\rho = \mathrm{Tr}_B|\psi\rangle\langle\psi| = CC^\dagger$ **?**

Spectral properties of random matrices

Non-hermitian matrix C of size N of the Ginibre ensemble



Under normalization $\mathrm{Tr}CC^\dagger = N$ the spectrum of C fills **uniformly** (for large N!) the **unit disk**

The so–called **circular law** of **Girko** !

Hermitian, positive matrix $\rho = CC^\dagger$ of the Wishart ensemble

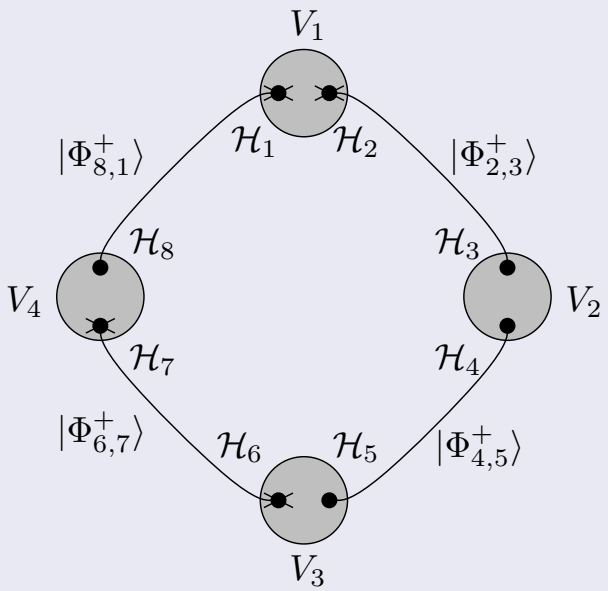
Let $x = N\lambda_i$, where $\{\lambda_i\}$ denotes the spectrum of ρ. As $\mathrm{Tr}\rho = 1$ so $\langle x \rangle = 1$. Distribution of the spectrum $P(x)$ is asymptotically given by the **Marchenko–Pastur law**

$$h_1(x) = P_{\mathrm{MP}}(x) = \frac{1}{2\pi}\sqrt{\frac{4}{x} - 1} \quad \text{for} \quad x \in [0, 4]$$

Spectral properties of random mixed states II

Example 2: 4 bonds, 8 subsystems and four bi-partite interactions V_i

c) h_2 random mixed state generated by the 4–**cycle graph**



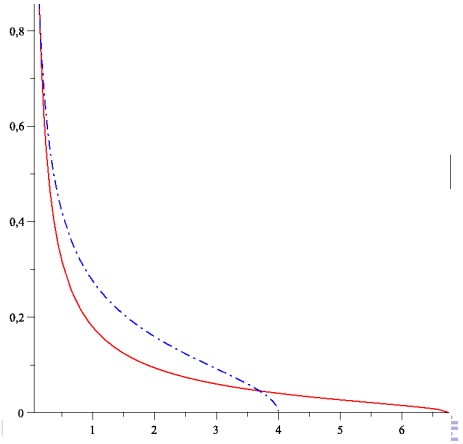
After partial trace over **crossed** subsystems the random mixed state has the structure

$$\rho = \alpha G_2 G_1 G_1^\dagger G_2^\dagger,$$

where G_1 and G_2 are independent **Ginibre** matrices and $\alpha = 1/\mathrm{Tr}\, G_2 G_1 G_1^\dagger G_2^\dagger$.

Mixed states with spectrum given by the **Fuss-Catalan distribution** $h_2(x)$ characterized by mean **entropy** $S(\rho) \approx \ln N - 5/6$

$P_{\mathbf{MP}}(x) = h_1(x)$ and $h_2(x)$.



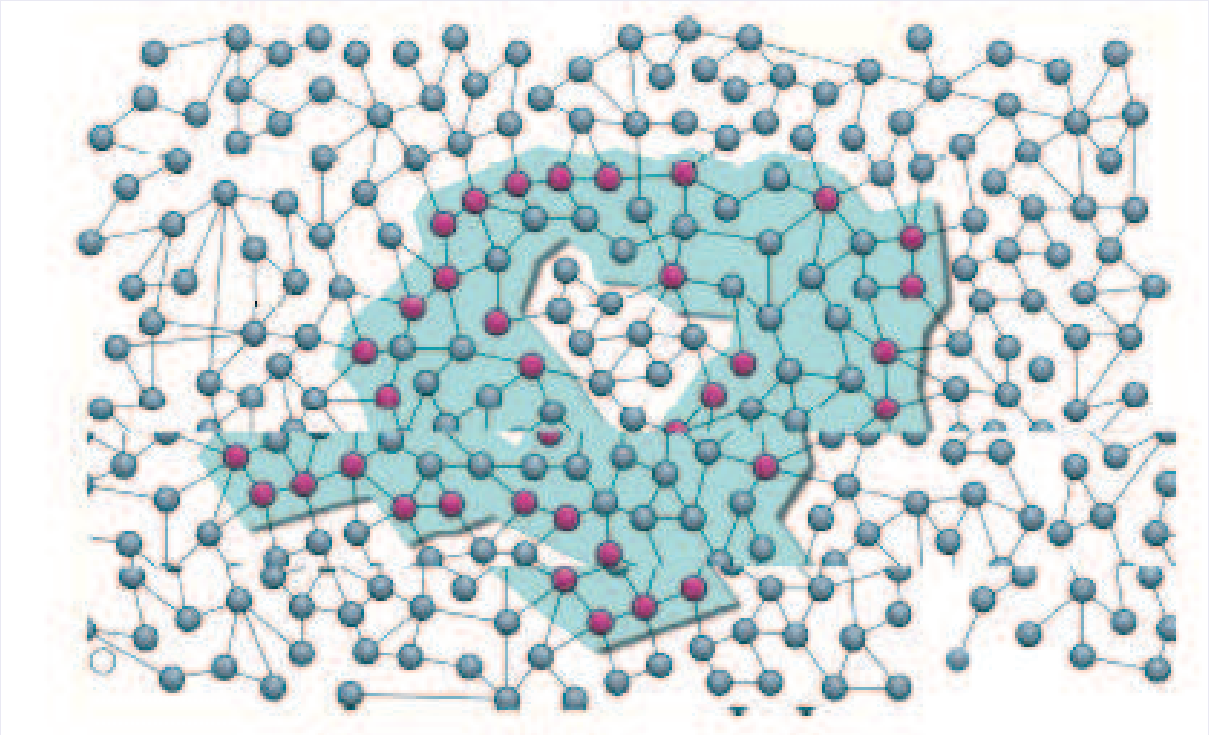
Multi–partite systems: a lattice L

Partition of the lattice into two disjoint sets, $L = A \cup \bar{A}$

Consider lattice (graph), in which each **vertex** denotes a spin

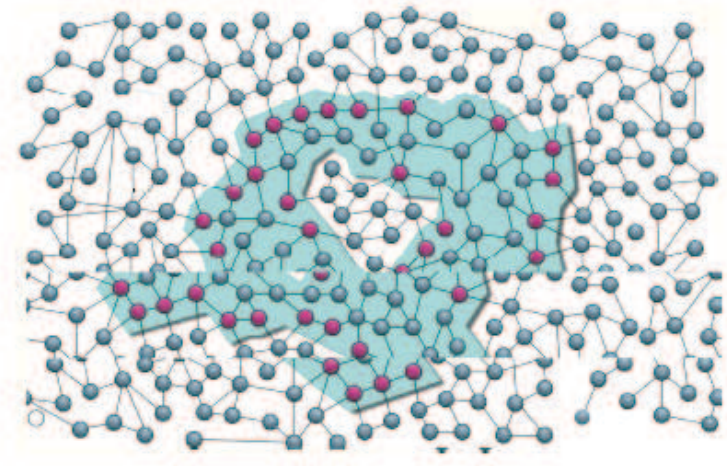
(*different meaning than before!*)

and each **edge** represents an interaction defined by a local Hamiltonian H.



Let A denotes a distinguished set of vertices while ∂A represents **spins** belonging to its **area**, i.e. these spins for which some edges are cut away.

Area law for a partition of the lattice $L = A \cup \bar{A}$



Consider an eigenstate $|\psi\rangle$ of the Hamiltonian H, define **set of spins** A and take the partial trace of the pure state over all spins belonging to the **complementary set** $\bar{A}$.

• **Von Neumann entropy** of the resulting mixed state $\rho := \mathrm{Tr}_{\bar{A}}|\psi\rangle\langle\psi|$ is proportional to the **area** ∂A of the distinguished subset A.
Hence **entanglement** of the state $|\psi\rangle$ with **respect to** the partition $A \cup \bar{A}$ behaves as the **area** ∂A.

Eisert, Cramer, Plenio 2008, Rev. Mod. Phys. 2008

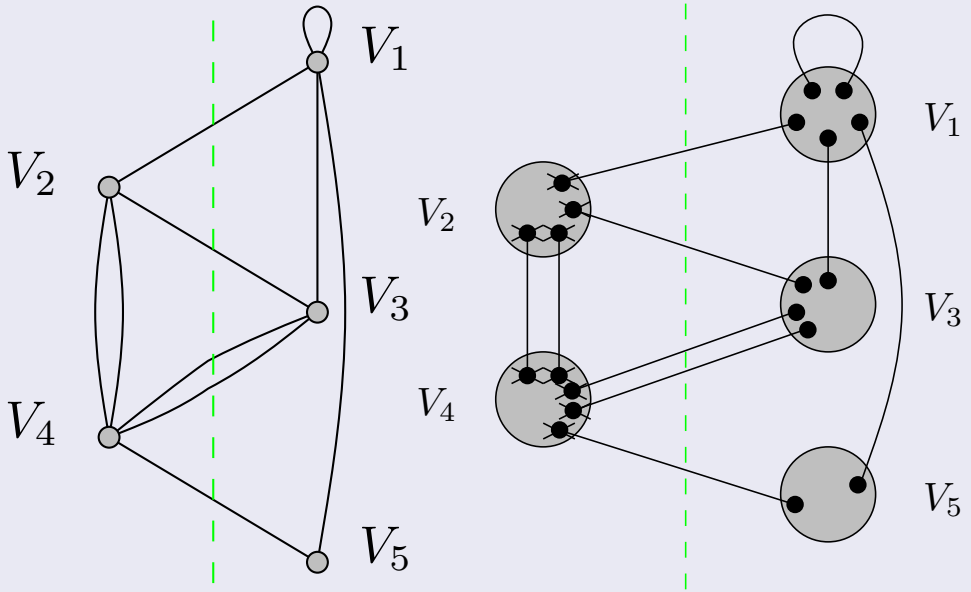
Universal Entanglement Area Law

Area law for random graph states

Theorem. Consider a graph Γ and its partition into two sets A and $\bar{A}$. Let $|\psi\rangle$ be a random graph pure state and $\rho := \mathrm{Tr}_{\bar{A}}|\psi\rangle\langle\psi|$.
Then the mean **entropy of** ρ (entanglement entropy of $|\psi\rangle$) is proportional to the **number** M of **bonds cut** ('area' of A) ,

$$\langle \mathbf{S}(\rho)\rangle_\psi = \mathbf{M}\ \ln \mathbf{N} + \mathbf{O(1)}\ .$$

Example: graph with 10 bonds, $M = 5$ of them cut



The **area law** $S(\rho) \approx 5 \ln N$ is **universal** – no dependence on the choice of interaction in the vertices.

Only the **topology** of the interaction matters!

B. Collins, I. Nechita, K. Ż., arXiv:1302.0709 (2013)

b) Structured random unitaries related to a graph

Two–step time evolution - for a system in $\mathcal{H}_{N^2 m}$

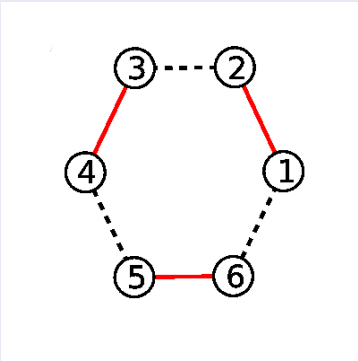
Let Γ be a graph consisting of k vertices $V_1, \ldots V_k$ and m bonds $B_1 \ldots B_m$. The corresponding **ensemble** of random unitary matrices of size N^{2m} reads

$$U \;=\; VW \;=\; \big(V_a \otimes \cdots \otimes V_k\big)\,\big(W_\alpha \otimes \cdots \otimes W_\mu\big),$$

where V_i and W_ν denote CUE random unitaries representing interaction in the **vertices** and along the **bonds**, respectively.

Attention: symbols $\otimes$ here and $\otimes$ here represent tensor product with respect to different spliting of the Hilbert space !

Example: triangle graph with $m = 3$ bonds and $k = 3$ vertices



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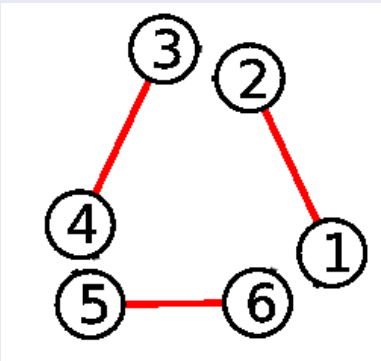


$$U_\triangle \;=\; VW \;=\; \big(V_{23} \otimes V_{45} \otimes V_{61}\big)\big(W_{12} \otimes W_{34} \otimes W_{56}\big)$$

Results: some properties of the graph ensembles

Level spacing distribution $P(s)$ for structured random unitaries

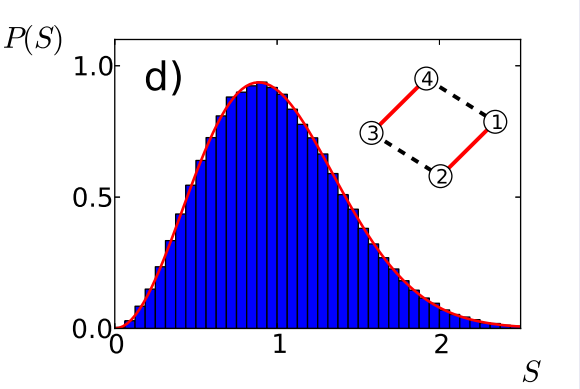
For **disconnected** graph: **Poissonian** statistics, $P(s) = \exp(-s)$

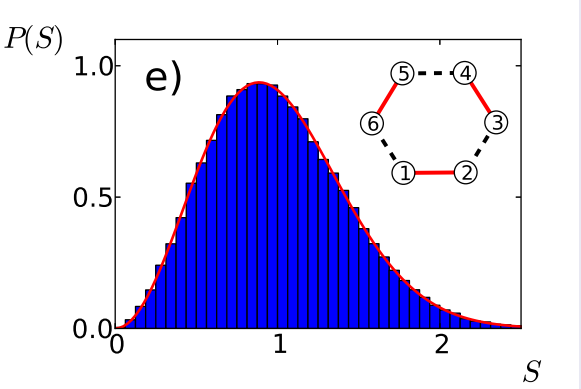
Explanation: for large dimension tensor product $U_1 \otimes U_2$ of two CUE matrices displays **Poissonian** level statistics

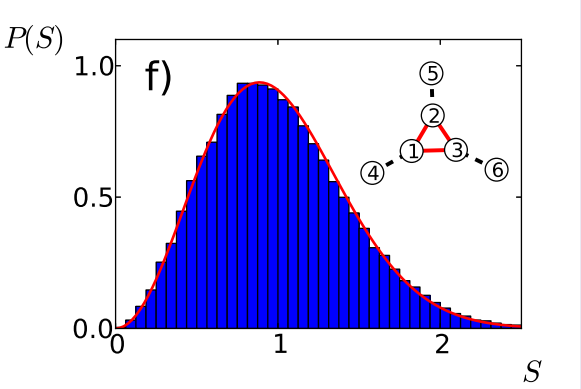
T. Tkocz, M. Smaczyński, M. Kuś, O. Zeituni, K.Życzkowski (2012)

For **connected** graph: **Wigner-like** CUE –like statistics

$$P(s) \approx \frac{32}{\pi^2} s^2 \exp(-\frac{4}{\pi} s^2)$$





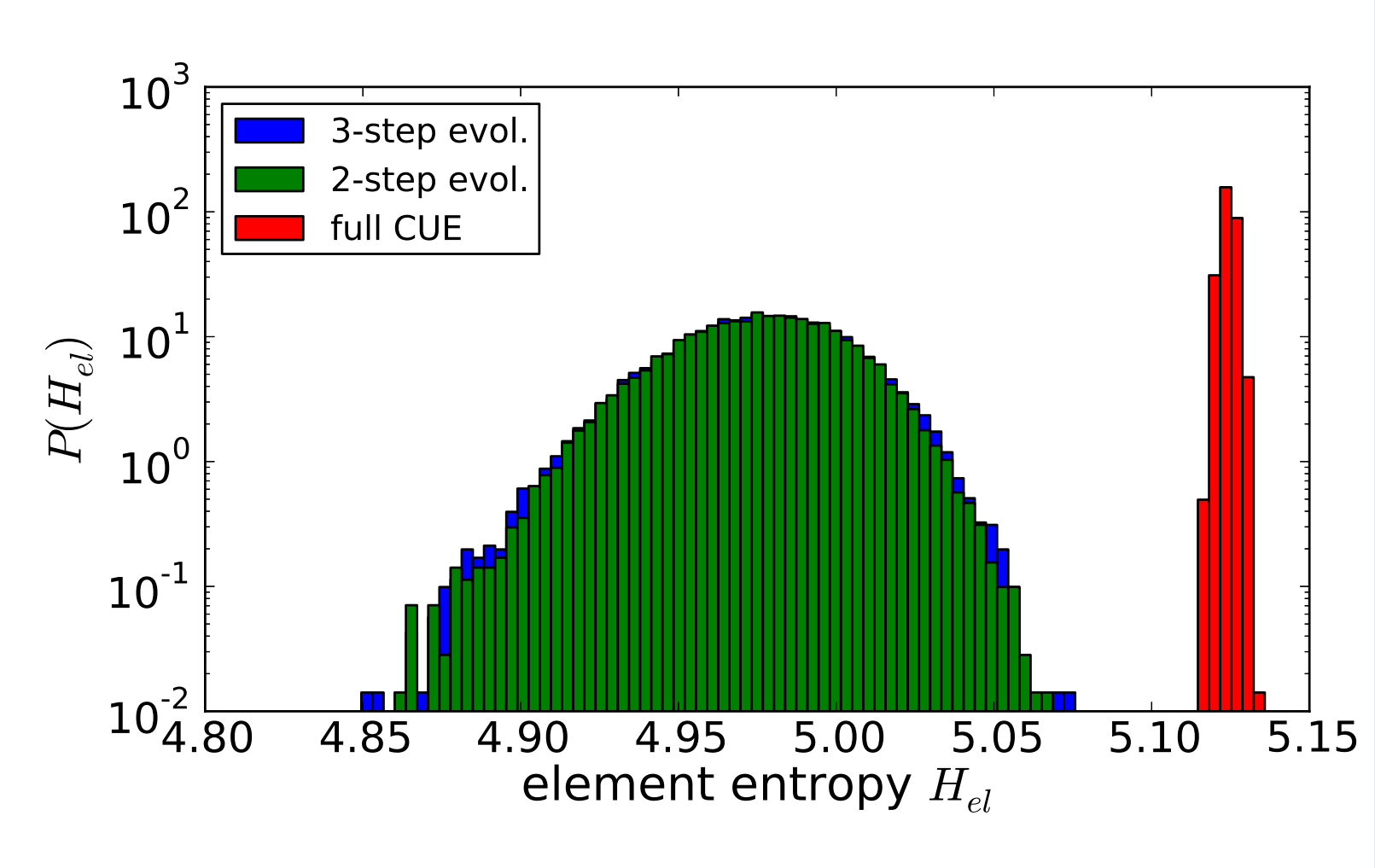


Distribution of matrix elements

Statistics of **matrix elements**, e.g. the elements **entropy**

$$H_{el}(U) \;:=\; -\frac{1}{M}\sum_{i=1}^{M}\sum_{j=1}^{M}|U_{ij}^2 \ln|U_{ij}|^2$$

differs from CUE matrices of size $M = N^{2m}$

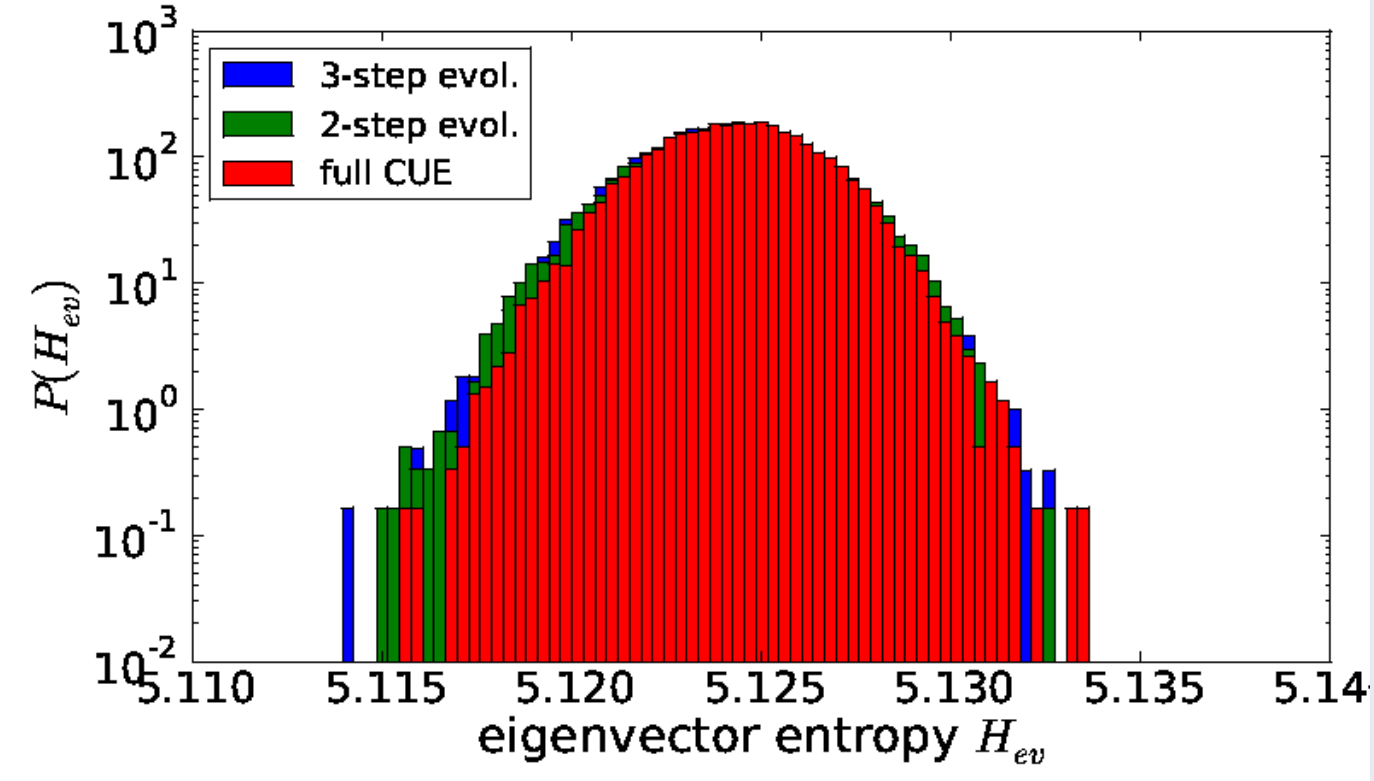


Distribution of eigenvectors

Statistics of **eigenvectors elements**, e.g. the eigenvector **entropy**

$$H_{el}(U) \;:=\; -\frac{1}{M}\sum_v \sum_{j=1}^{M} |v_j|^2 \ln |v_j|^2$$

behaves like for **CUE random matrices** of size $M = N^{2m}$



"Cheap" method to generate random CUE matrices of size $M = n^k$ out of k smaller CUE matrices of size n^2

$$U \;=\; VW \;=\; \left(V_{23} \otimes V_{45} \otimes \cdots \otimes V_{k,1}\right)\left(W_{12} \otimes W_{34} \otimes \cdots \otimes W_{k-1,k}\right).$$

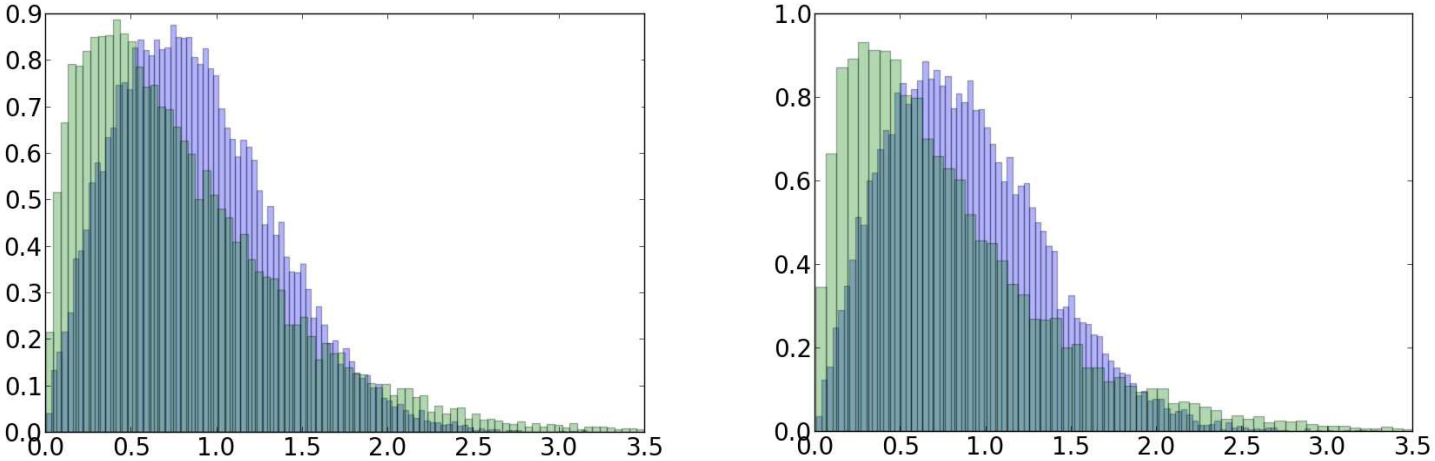
Topology of the graph and corresponding ensembles of random matrices

Search for statistical properties of ensembles depending on the topology of the graph

Motivation:

Isomorphic graphs problem:

For two (classical) graphs Γ_1 and Γ_2, with the same number of vertices and bonds, check, whether these graphs are **isomorphic**, $\Gamma_1 \sim \Gamma_2$?



Distributions $P(\langle 1|U_1^\dagger U_2|1\rangle|)$ and $P(|\mathrm{Tr}\, U_1^\dagger U_2|)$ for random matrices U_1 and U_2 from two ensembles associated to different graphs.

Concluding remarks

- A **quantized chaotic evolution** sends an initial state $|i\rangle$ into a **'generic'** $=$ **random** state $|\psi\rangle$.
- Due to **concentration of measure** effect (**Levy's lemma**) a value of a function f of a random state is close to the **mean value** $\langle f \rangle$.
- Generic pure state of a **bi-partite system** is strongly **entangled**.
- With any graph Γ we associate an ensemble of **a) random pure states** and **b) random unitary matrices**.
- **Case a)**. Performing **partial trace** over a selected set A of subsystems one obtains an ensemble of mixed states.

  Their **level density** is related to the topology of the graph.
- **Universal Entanglement Area law**: For any **graph** Γ and its partition A and $\bar{A}$ he **mean entanglement entropy** of the **random pure state** $|\psi\rangle$ depends on **the area** ∂A (the number of **bonds cut**).
- **Case b)**. Level spacing statistics $P(s)$ for ensembles of **random unitary** matrices associated to a connected graph has CUE properties.
- More subtle statistics do depend on **graph topology**.

  Potential application: graph isomorphism problem.