N-particle quantum statistics on graphs

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Quantum statistics

- One of the postulates of non-relativistic quantum mechanics says:
- For n indistinguishable particles the Hilbert space is

$$\mathcal{H}_F = \bigwedge^n (\mathcal{H}_1)$$

$$\mathcal{H}_B = \mathrm{Sym}^n (\mathcal{H}_1)$$

- In terms of the wave function when two fermions are exchanged the sign of the wave function changes and for bosons it stays the same
- It was first noticed by Souriau, and subsequently by Leinaas and Myrheim that this additional postulate can be understood in terms of topological properties of the classical configuration space of indistinguishable particles.

Configuration spaces

- M – the one-particle classical configuration space (e.g., an m-dimensional manifold)
- $F_n(M)$ – the space of n distinct points in M.
$$F_n(M) = \{(x_1,\, x_2, \ldots,\, x_n)\ : x_i \in X,\, x_i \neq x_j\} = M^{\times n} - \Delta$$
$$\Delta = \{(x_1, x_2, \ldots, x_n) : \exists_{i,j}\, x_i = x_j\}$$
- The n-particle configuration space is defined as an orbit space
$$C_n(M) = {}^{F_n(M)}/_{S_n},$$
where S_n is the permutation group of n elements and the action of S_n on $F_n(M)$ is given by
$$\sigma(x_1\,, \ldots\, x_2) = (x_{\sigma(1)}\,, \ldots\, x_{\sigma(2)}), \quad \forall \sigma \in S_n.$$
- Any closed loop in $C_n(M)$ represents a process in which particles start at some particular configuration and end up in the same configuration modulo that they might have been exchanged.

Bosons and fermions in $\mathbb{R}^3$

- The abelianization of $\pi_1(C_n(M))$ is the first homology group $H_1(C_n(M))$
- Souriau, Leinaas and Myrheim: $H_1(C_n(M))$ encodes information about quantum statistics.
- Example1: Bosons and Fermions on $M = \mathbb{R}^m$, where $m \geq 3$
- The fundamental group

$$\pi_1(C_n(\mathbb{R}^m)) = S_n$$

- The first homology

$$H_1(C_n(\mathbb{R}^m)) = \mathbb{Z}_2 = (\{1\,,\, e^{i\pi}\}, \cdot)$$

Anyons in $\mathbb{R}^2$

- Example 2: Anyons on $M = \mathbb{R}^2$
- $\pi_1(C_n(\mathbb{R}^2))$ is Artin braid group $\mathrm{Br}_n(\mathbb{R}^2)$

$$\mathrm{Br}_n(\mathbb{R}^2) = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \,|\, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i \sigma_j = \sigma_j \sigma_i \rangle,$$

  where in the first group of relations we take $1 \leq i \leq n-2$, and in the second, we take $|i-j| \geq 2$.

- The abelianization of $\pi_1(C_n(\mathbb{R}^2))$ is

$$H_1(C_n(\mathbb{R}^2)\,,\,\mathbb{Z}) = \mathbb{Z}.$$

- Any phase is possible for particles exchange

Quantum statistics on graphs

- $\Gamma = (V, E)$ be a metric connected simple graph on $|V|$ vertices and $|E|$ edges.
- Similarly to the previous cases we define

$$F_n(\Gamma) = \Gamma^{\times n} - \Delta$$

$$\Delta = \{(x_1, x_2, \ldots, x_n) : \exists_{i,j}\, x_i = x_j\}$$

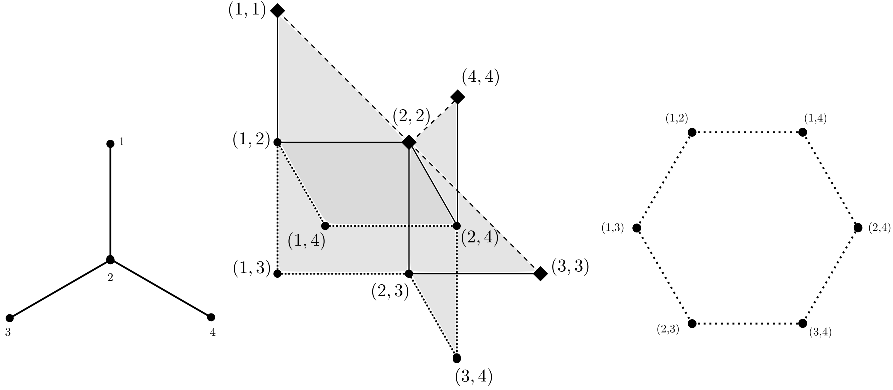
  and

$$C_n(\Gamma) = F_n(\Gamma)/S_n,$$

- The calculation of the first homology $H_1(C_n(\Gamma))$ of $C_n(\Gamma)$ can be reduced to the computation of the first homology of some simpler space.
- n-particle combinatorial configuration space as

$$\mathcal{D}^n(\Gamma) = (\Gamma^{\times n} - \tilde{\Delta})/S_n,$$

  where $\tilde{\Delta}$ denotes all cells whose closure intersects with Δ.



(a) (b) (c)

Combinatorial configuration spaces

- Abrams (2000): For any graph Γ with at least n vertices, the inclusion $\mathcal{D}^n(\Gamma) \hookrightarrow C_n(\Gamma)$ is a homotopy equivalence iff the following hold:
  - Each path between distinct vertices of valence not equal to two passes through at least $n-1$ edges.
  - Each closed path in Γ passes through at least $n+1$ edges.
- Conclusion: $H_1(\mathcal{D}^n(\Gamma)) = H_1(\mathcal{C}_n(\Gamma))$
- Forman (1998) - discrete Morse theory - a general method for calculation $H_k(X)$, where X - CW complex
- Farley, Sabalka (2005) - $H_1(\mathcal{D}^n(\Gamma))$ for tree graphs
- Ko and Park (2011) - the full description of $H_1(\mathcal{D}^n(\Gamma))$ - highly technical
- AS (2012) - an alternative formulation of discrete Morse theory for $\mathcal{D}^n(\Gamma)$
- Our goal: An elementary proof of the structure theorem for $H_1(\mathcal{D}^n(\Gamma))$ - not using (or using in the limited way) discrete Morse theory

Homology of finitely generated modules

- By the structure theorem for finitely generated modules:

$$H_1(\mathcal{D}^n(\Gamma)) = \mathbb{Z}^k \oplus T_l$$

  where T_l is the torsion, i.e.

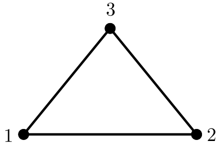
$$T_l = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_l},$$

  and $n_i | n_{i+1}$.
- In other words $H_1(\mathcal{D}^n(\Gamma))$ is determined by k free parameters $\{\phi_1, \ldots, \phi_k\}$ and l discrete parameters $\{\psi_1, \ldots, \psi_l\}$ such that for each $i \in \{1, \ldots l\}$

$$n_i \psi_i = 0 \text{ mod } 2\pi, \ \ n_i \in \mathbb{N} \ \ \text{and } n_i | n_{i+1}.$$

- We will call the parameters ϕ and ψ continuous and discrete phases respectively.

Examples



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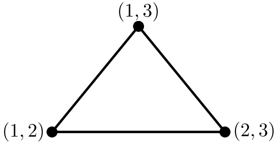

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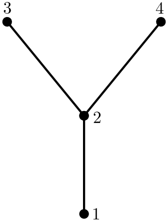

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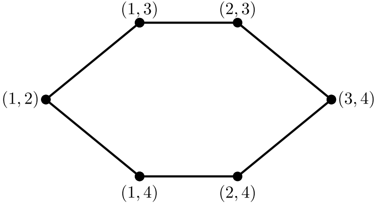

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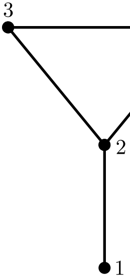
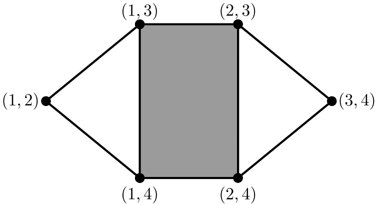
Examples



(a)

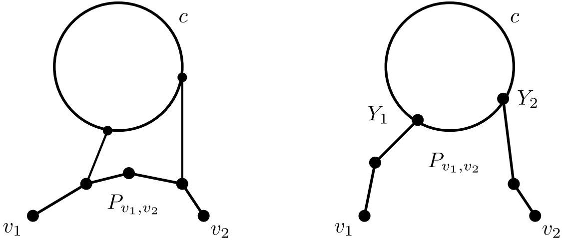


(b)

$$\phi_{c,2} = \phi_{c,1}^{1} + \phi_Y.$$

Knowing ϕ_Y and the AB-phase determines the phase $\phi_{c,2}$.

An overcomplete spanning set for $H_1(\mathcal{D}^2(\Gamma))$



(a)

(b)

$$\phi_{c,2} = \phi_{c,1}^{v_1} + \phi_{Y_1}, \quad \phi_{c,2} = \phi_{c,1}^{v_2} + \phi_{Y_2},$$

and hence

$$\phi_{c,1}^{v_1} - \phi_{c,1}^{v_2} = \phi_{Y_2} - \phi_{Y_1}.$$

The relations between different AB-phases for a fixed cycle c of Γ are encoded in the phases ϕ_Y.

An overcomplete spanning set for $H_1(\mathcal{D}^2(\Gamma))$

We will use a spanning set containing the following:

1. All 2-particle cycles corresponding to the exchanges on Y subgraphs of Γ. There can be dependencies between these cycles.
2. A set of $\beta_1(\Gamma) = E - V + 1$ AB-cycles, one for each independent cycle in Γ.

- Thus, $H_1(\mathcal{D}^2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$, where A is determined by Y-cycles.
- Consequently, in order to determine $H_1(\mathcal{D}^2(\Gamma))$ one has to study the relations between Y-cycles.

3-connected graphs

- Γ is n-connected graph if there are at least n internally disjoint paths between any pair of vertices of Γ.
- The basic examples of 3-connected graphs are wheel graphs.
- (Wheel theorem) Let Γ be a simple 3-connected graph different from a wheel. Then for some edge $e \in E(\Gamma)$ either $\Gamma \setminus e$ or Γ / e is simple and 3-connected.
- $\Gamma \setminus e$ is constructed from Γ by removing the edge e – edge removal.
- Γ / e is obtained by contracting edge e and identifying its vertices – edge contraction.
- The inverses will be called edge addition and vertex expansion.
- Any simple 3-connected graph can be constructed in a finite number of steps starting from a wheel graph W^k, for some k

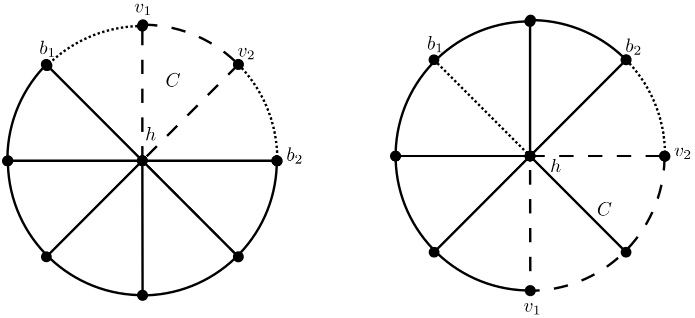
$$W_k = \Gamma_0 \mapsto \Gamma_1 \mapsto \ldots \mapsto \Gamma_{n-1} \mapsto \Gamma_n = \Gamma$$

  where Γ_i is constructed from Γ_{i-1} by either
  1. Adding an edge between non-adjacent vertices or
  2. Expanding at the vertex of the valency at least four.

  Moreover, each Γ_i is simple and 3-connected.

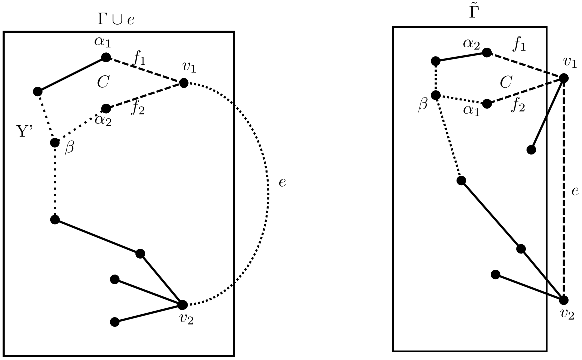
3-connected graphs

- In order to prove inductively some feature of a 3-connected graph it is enough to show it for an arbitrary wheel and consider what happens when an edge between two non-adjacent vertices is added or a vertex of valency at least four is expanded.
- Lemma 1: For wheel graphs W^n all phases ϕ_Y are equal up to the sign.



(a) (b)

- Lemma 2: For 3-connected simple graphs all phases ϕ_Y are equal up to the sign.



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3-connected graphs

- Theorem 1: For a 3-connected simple graph, $H_1(\mathcal{D}^2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$, where $A = \mathbb{Z}_2$ for non-planar graphs and $A = \mathbb{Z}$ for planar graphs.
- Proof: By Lemmas 1 and 2 we only need to determine the phase ϕ_Y.
- By direct calculation for the graphs K_5 and $K_{3,3}$, $H_1(\mathcal{D}^2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus \mathbb{Z}_2$, i.e. $\phi_Y = 0$ or π.
- Kuratowski's theorem: Every non-planar graph contains a subgraph which is isomorphic to K_5 or $K_{3,3}$.
- This proves the statement for non-planar graphs.
- For planar graphs we have the anyon phase and hence $A = \mathbb{Z}$.