On spectral gap for quantum graphs

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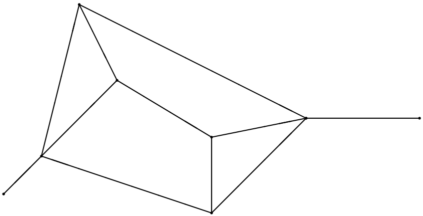
May 26, 2013

Chaos 006

Quantum graph

- **Metric graph**



- **Differential expression** on the edges

$$\ell_{q,a} = \left(i \frac{d}{dx} + a(x) \right)^2 + q(x)$$

- **Matching conditions**
  Via irreducible unitary matrices S^m associated with each internal vertex V_m

$$i(S^m - I)\vec{\psi}_m = (S^m + I)\partial\vec{\psi}_m, \;\; m = 1, 2, \dots, M.$$

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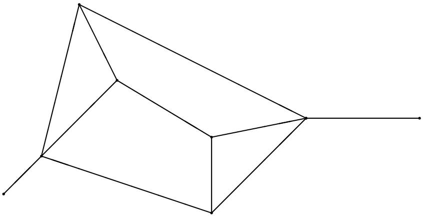

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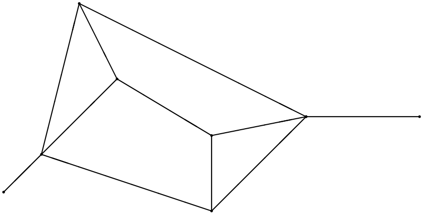

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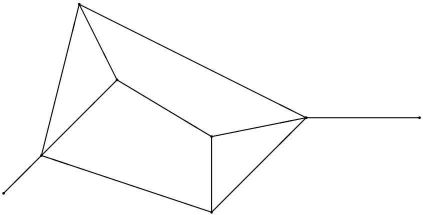
Standard Laplacian

- **Metric graph**



- **Differential expression** on the edges

$$\ell_{q,a} = -\frac{d^2}{dx^2}$$

- **Standard matching conditions**

$$\begin{cases} \text{the function is continuous at } V_m, \\ \text{the sum of normal derivatives is zero.} \end{cases}$$

Spectral properties

Our assumption:

- The metric graph Γ is connected and formed by a finite number of compact edges.

Properties:

The spectrum is discrete and tends to ∞ satisfying Weyl's asymptotic law

$$\lambda_n \sim \left(\frac{\pi}{\mathcal{L}}\right) n^2$$

$\lambda_0 = 0$ is a simple eigenvalue with the eigenfunction $\psi_0 = 1$.

The asymptotic behavior of the spectrum determines the Euler characteristics of the underlying metric graph.

The spectral gap $\lambda_1 - \lambda_0$ coincides with λ_1.

Discrete graphs:

M. Fiedler: the spectral gap for **discrete Laplacians** is the measure of their connectivity $\Rightarrow \lambda_1$ is called the **algebraic connectivity**.

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Our questions today

- How does spectral gap depend on connectivity?
- What happens to the spectral gap as new edges are added to a graph?
- Which graphs have the smallest spectral gap?
- Explicit estimates for the spectral gap?

What was known?

- The spectral gap can be calculated using Rayleigh quotient

$$\lambda_1(\Gamma) = \min_{\substack{\psi \perp 1 \\ \psi \in C(\Gamma)}} \frac{\int_\Gamma |\psi'|^2 dx}{\int_\Gamma |\psi|^2 dx}. \tag{1}$$

- Joining two vertices into one increases the spectral gap, since the number of admissible functions in (1) decreases.
- L. Friedlander (2005): among all graphs with the same total length single interval-graph has the smallest spectral gap
  - We learned about the proof after our paper was accomplished
  - The proof uses symmetrization technique and is irrelevant to the topology of the underlying graph
    "*First it is sufficient to prove the inequality ... for trees*" (L. Friedlander)
- P. Exner and M. Jex (2012) demonstrated unusual behavior of the spectral gap for Quantum graphs with delta interactions at the vertices.

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Estimate for the spectral gap

Theorem 1. *(Friedlander, Ku-Naboko)*

$$\lambda_1(\Gamma) \geq \lambda_1(\Delta_{\mathcal{L}(\Gamma)}) = \left(\frac{\pi}{\mathcal{L}(\Gamma)}\right)^2.$$

Proof

$$\lambda_1 = \frac{\int_\Gamma |\psi_1'|^2 dx}{\int_\Gamma |\psi_1|^2 dx}$$

ψ_1 - the eigenfunction corresponding to λ_1
$\Gamma \Rightarrow \Gamma^2$ - double cover of Γ - the graph with the same set of vertices but with all edges doubled

$$\lambda_1 = \frac{\int_{\Gamma^2} |\psi_1'|^2 dx}{\int_{\Gamma^2} |\psi_1|^2 dx}$$

Γ^2 is a graph with even degrees of all vertices $\Rightarrow$ there exists Eulerian path going along every edge precisely one time. This path can be identified with the loop of double length $S_{2\mathcal{L}}$.

$$\lambda_1 \geq \min \frac{\int_{S_{2\mathcal{L}}} |u'|^2 dx}{\int_{S_{2\mathcal{L}}} |u|^2 dx} = \lambda_1(S_{2\mathcal{L}}) = \lambda_1(\Delta_{\mathcal{L}}).$$

Corollary 1
Among all graphs of the same total length single interval has the smallest spectral gap.
Geometric version of Ambartsumian theorem (1929).

Corollary 2
Similar estimates for all higher eigenavlues.

Corollary 3 (Ku-Naboko)
If the degrees of all vertices in Γ are even, then the following estimate holds

$$\lambda_1(\Gamma) \geq \lambda_1(S_{\mathcal{L}}) = 4\lambda_1(\Delta_{\mathcal{L}(\Gamma)}) = 4\left(\frac{\pi}{\mathcal{L}(\Gamma)}\right)^2.$$

Proof: no need to double the edges
Cannot be proven using symmetrization technique.

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Adding an edge

Adding an edge increases the total volume of the graph $\Rightarrow$ the first eigenvalue has a tendency to decrease (in contrast to discrete graphs!).

Adding pending edge
Theorem

1. The first eigenvalues satisfy the following inequality:

$$\lambda_1(\Gamma) \geq \lambda_1(\Gamma').$$

2. The equality $\lambda_1(\Gamma) = \lambda_1(\Gamma')$ holds if and only if every eigenfunction ψ_1 corresponding to $\lambda_1(\Gamma)$ is equal to zero at V_1

$$\psi_1(V_1) = 0.$$

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Adding an edge between two exciting vertices

Two different tendencies:

- Adding an edge increases the volume $\Rightarrow \lambda_1 \searrow$
- Connecting two vertices introduces new restrictions $\Rightarrow \lambda_1 \nearrow$

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If the eigenfunction ψ_1 corresponding to the first excited eigenvalue can be chosen such that

$$\psi_1(V_1) = \psi_1(V_2),$$

then:

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Theorem 2

If the length of the new edge is greater than the total length of the original graph, then the spectral gap decreases.

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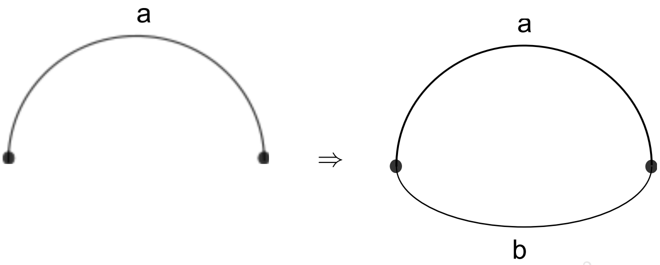

$$\lambda_1 = \left(\frac{\pi}{a}\right)^2 \qquad \lambda_1 = \left(\frac{2\pi}{a+b}\right)^2$$

Conclusions:

$$\begin{cases} a < b & \Rightarrow & \lambda_1 \text{ decreases} \\ a > b & \Rightarrow & \lambda_1 \text{ increases} \end{cases}$$

Theorem 2 is sharp!

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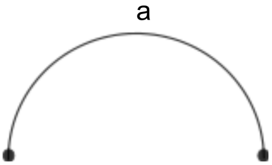

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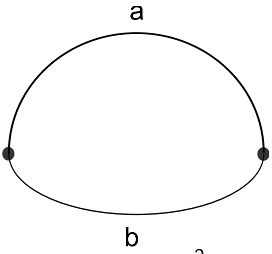

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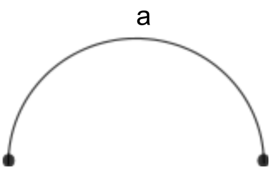

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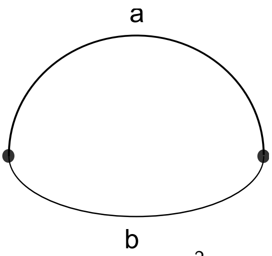

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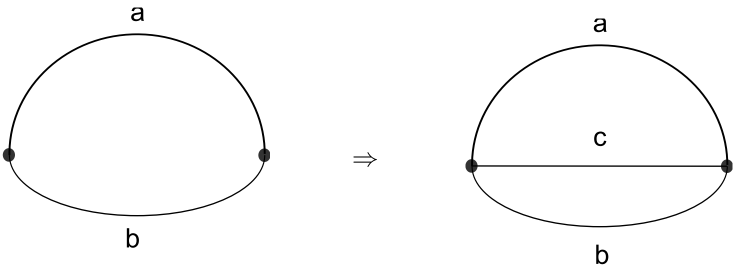

The spectral gap always decreases!

Theorem 1 works here!

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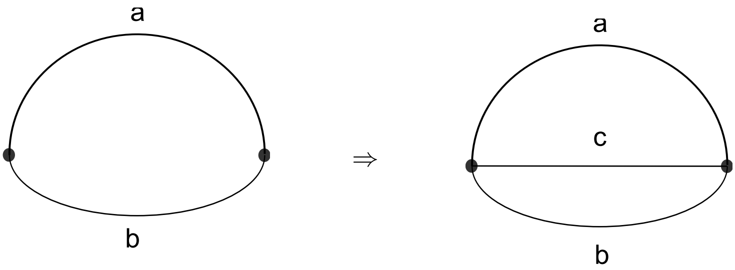

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Cutting and deleting edges

Two tendencies as before:

- Cutting an edge removes certain connections $\Rightarrow \lambda_1 \searrow$
- Deleting an edge or a piece of it reduces the volume $\Rightarrow \lambda_1 \nearrow$

Theorem

Γ - graph of length $\mathcal{L}$. We cut of a piece from the edge connecting V_1 and V_2. If

$$\left(\max_{\psi_1 : L^{\rm st}(\Gamma)\psi_1 = \lambda_1 \psi_1} \frac{(\psi_1(V_1) - \psi_1(V_2))^2}{(\psi_1(V_1) + \psi_1(V_2))^2} \cot^2 \frac{k_1 \ell}{2} - 1 \right) \frac{k_1}{2} \cot \frac{k_1 \ell}{2} \geq (\mathcal{L} - \ell)^{-1},$$

then

$$\lambda_1(\Gamma^*) \leq \lambda_1(\Gamma).$$

One can estimate the length of a piece that can be cut away from the edge so that the spectral gap decreases:

$$\psi_1(x) = \alpha \sin k_1(x - x^*) + \beta \cos k_1(x - x^*) \Rightarrow$$

$$\ell \leq \min \left\{ \frac{\pi}{2k_1}, \ \frac{\pi}{4}(\mathcal{L} - \ell) \max_{\psi_1 : L^{\rm st}(\Gamma)\psi_1 = \lambda_1 \psi_1} \left(\frac{\alpha^2}{\beta^2} - 1 \right) \right\}.$$

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