Fading Statistics in Communications –
A Random Matrix Approach

Jen-Hao Yeh, Edward Ott, Thomas Antonsen, Steven M. Anlage

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Motivation

Wave propagation in a complex environment

With the help of Random Matrix Theory (RMT),

Wave-chaotic systems are expected to show universal statistical properties.

Casati, Valz-Gris, Guarnieri, (1980)
Bohigas, Giannoni, Schmidt, PRL (1984)

Are there any new phenomena to investigate?
Fading

- Radiowave propagation through wireless channels is a complicated phenomenon characterized by various effects such as multipath and shadowing.
Radiowave propagation through wireless channels is a complicated phenomenon characterized by various effects such as multipath and shadowing.
Fading in Communications

Multi-path Propagation

Reflection with partial cancellation

Knife-edge Diffraction

“Rain Fades” on Microwave Links
Apply \textit{wave chaos theory} to derive a physical model for the fading phenomenon.

\textbf{Verify the model with new experiments.}

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{Semi-empirical statistical models} & \textbf{First-principles-derived model} \\
\hline
Model A & \textbf{Model A} & \textbf{Model B} & \textbf{Model C} & Model … \\
Parameter & $a_1$ & Parameters & Parameters & Parameters \\
(b_1, b_2) & & (c_1, c_2, c_3) & & \\
\hline
\end{tabular}
\end{center}

\textbf{Different models have been proposed for different environments and different wave propagation applications …}
Outline

• What is Fading?

• More About Fading

• A Simple RMT Model of Fading

• Experiments to Test the Fading Model

• Prediction of Fading Amplitudes

• Conclusions
Fading due to 3 people walking around the receiving antenna within a 2 meter radius

5 GHz ($\lambda = 6$ cm) Unlicensed-National Information Infrastructure (U-NII) wireless signal

Ionospheric Fading
Global Positioning System (GPS) Signal Propagation

Data collected at Ascension Island in the South Atlantic Ocean, March, 2001 during the solar maximum

CHARACTERISTICS OF DEEP GPS SIGNAL FADING DUE TO IONOSPHERIC SCINTILLATION FOR AVIATION RECEIVER DESIGN

Jiwon Seo, Todd Walter, Tsung-Yu Chiou, and Per Enge
Stanford University
Fading models

Rayleigh fading

without the line-of-sight signal

Rician fading

with a strong line-of-sight signal
Semi-Empirical Fading Models

Fading amplitude \( R = \frac{\text{Received amplitude}}{\text{Transmitted amplitude}} = \left| S_{21} \right| \)

Rayleigh fading

(one parameter) \( P(R; \sigma) = \frac{R}{\sigma^2} e^{-R^2/2\sigma^2} \)

\( \sigma = 0.1 \)
\( \sigma = 0.2 \)
\( \sigma = 0.3 \)

Rician fading

(two parameters) \( P(R; \sigma, \nu) = \frac{R}{\sigma^2} e^{-(R^2/2\sigma^2)} I_0 \left( \frac{R \nu}{\sigma^2} \right) \)

\( \sigma = 0.1 \)
\( \nu = 0.1 \)
\( \nu = 0.3 \)
\( \nu = 0.5 \)
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N-Port Description of an Arbitrary Scattering System

N Ports

- Voltages and Currents,
- Incoming and Outgoing Waves

\[ K = -i \, Z \]

Reaction Matrix

**S matrix**

\[
\begin{bmatrix}
V_1^- \\
V_2^- \\
\vdots \\
V_N^-
\end{bmatrix}
= [S] \cdot
\begin{bmatrix}
V_1^+ \\
V_2^+ \\
\vdots \\
V_N^+
\end{bmatrix}
\]

**Z matrix**

\[
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_N
\end{bmatrix}
= [Z] \cdot
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_N
\end{bmatrix}
\]

\[ S = (Z + Z_0)^{-1}(Z - Z_0) \]

\[ Z(\omega), \, S(\omega) \]

- We are concerned mainly with 2-port systems
The Scattering Matrix


Polar Decomposition of perfectly coupled $s$: $S = u \left( \begin{array}{cc} \sqrt{1-T_1} & 0 \\ 0 & \sqrt{1-T_2} \end{array} \right) u'$.

$u$ is a random unitary matrix

$T_1$ and $T_2$ give the “absorption strength” of the fictitious lead (quantum transport)

$P(T_1, T_2) = \frac{1}{8} T_1^{-4} T_2^{-4} \exp\left[ -\frac{1}{2} \gamma (T_1^{-1} + T_2^{-1}) \right] |T_1$

$-T_2]\left[ \gamma^2 (2 - 2e^\gamma + \gamma + \gamma e^\gamma) \\
-\gamma (T_1 + T_2)(6 - 6e^\gamma + 4\gamma + 2\gamma e^\gamma + \gamma^2) \\
+ T_1 T_2(24 - 24e^\gamma + 18\gamma + 6\gamma e^\gamma + 6\gamma^2 + \gamma^3) \right]$

Joint distribution for $T_1$ and $T_2$ for $\beta = 1$

One mode in each lead

$0 \leq T_1, T_2 \leq 1$

Statistics depend only on the de-phasing (loss) parameter $\gamma$ ($\alpha$). $\gamma = 4\pi \alpha$
Statistical Fluctuations in Eigenvalues of $SS^\dagger$

S. Hemmady, et al., PHYSICAL REVIEW E 74, 036213 (2006)

Loss Case 0:
Frequency 3.2-4.2 GHz

$\gamma_{MLE} = 12.4$
$\alpha \approx 1.0$

Loss Case 0:
Frequency 13.5-14.5 GHz

$\gamma_{MLE} = 36.5$
$\alpha \approx 2.9$

Theoretical Contours obtained from Beenakker and Brouwer PRB 55,4695 (1997)
Microwave Billiards Mimic a 2-Dimensional Quantum Dot

Uniformly-distributed microwave losses are equivalent to quantum “de-phasing”

Brouwer+Beenakker (1997)

### Microwave Losses

<table>
<thead>
<tr>
<th>Loss Parameter:</th>
<th>Quantum De-Phasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = \frac{3dB \text{ bandwidth}}{\Delta \omega}$</td>
<td>$\gamma = 0$ Pure quantum transport</td>
</tr>
<tr>
<td>$\alpha = \frac{\text{Mean spacing}}{\Delta \omega}$</td>
<td>$\gamma \rightarrow \infty$ Classical limit</td>
</tr>
</tbody>
</table>

By comparing the Poynting theorem for a cavity with uniform losses to the continuity equation for probability density, one finds:

$$4\pi \alpha \leftrightarrow \gamma$$

### Determination of $\gamma$

- $x = 0$ : Loss
- $x = 16$ : Loss
- $x = 32$ : Loss

**Data**

- $\gamma \approx 10$
- $\approx 10^5$ data points

**Theory**

- $\gamma \approx 10$
- $\approx 10^5$ data points

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**Determination of $\gamma$**

**PDF(eigenvalues of $SS$)**

$\gamma(T_1, T_2)$

Brouwer+Beenakker, PRB (1997)
Prediction for the Fading Amplitude

\[ |S_{21}| = \sqrt{\left(\xi - \xi^2\right)(2 - T_1 - T_2 - 2\sqrt{(1 - T_1)(1 - T_2)} \cos \theta)} \]

where

\[ \begin{cases} 
\xi & \text{is uniformly distributed in } [0, 1) \\
\theta & \text{is uniformly distributed in } [0, 2\pi) 
\end{cases} \]

... and come from the random unitary matrix

The fluctuations of the fading amplitude are governed by a single parameter: the loss parameter \( \alpha \)

**In the limit of high loss:**

\[ P\left(|S_{21}|; \alpha \to \infty \right) = 8\pi\alpha |S_{21}| e^{-4\pi\alpha |S_{21}|^2} \]

This is the Rayleigh distribution, widely used to describe fading in communications:

\[ P\left(|S_{21}|; \sigma \right) = \frac{|S_{21}|}{\sigma^2} e^{-|S_{21}|^2 / 2\sigma^2} \]

Hence \( \alpha = \frac{1}{8\pi\sigma^2} \)
Comparison of Fading Amplitude Predictions

**Graph Description:**

- **Axes:**
  - Y-axis: Probability density
  - X-axis: $|S_{21}|$

- **Curves:**
  - **RMT model ($\alpha = 0$):** Green
  - **RMT model ($\alpha = 0.1$):** Blue
  - **RMT model ($\alpha = 1$):** Cyan
  - **RMT model ($\alpha = 10$):** Red

- **Distribution:**
  - **Rayleigh distribution:** Dashed black line

**Legend:**
- RMT model ($\alpha = 10$)
- Rayleigh distribution
- RMT model ($\alpha = 1$)
- RMT model ($\alpha = 0.1$)
- RMT model ($\alpha = 0$)
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The Microwave Billiard Experiment: A simplified model of wave-chaotic scattering systems

A thin hollow metal box

Coaxial Cable
Z₀ = 50 Ω

Side view

Re[Z₁]

Resistance (Ohm)

Frequency (GHz)
Ensemble to Simulate the Fading Environment

Perturbers change the boundaries
This is a single realization.
Ensemble to Simulate the Fading Environment

Another realization

```
\text{Re}[Z_{11}]
```
Ensemble to Simulate the Fading Environment

Keep doing this 100 times → Yet another realization
The Random Coupling Model Removes Non-chaotic Features in the Z / S Data

To compare data to RMT predictions, we must remove the non-chaotic features in the data …

1. Radiation Properties of Antennas
2. Major (short) trajectories
   long trajectories $\rightarrow$ chaotic

Extended Random Coupling Model


“A theory of the finite ensemble”
The Extended Random Coupling Model
Non-Universal Wave Scattering Properties

1-Port, Loss-less case:

\[ Z = iX_{avg} + R_{avg}i\xi \]

Universally Fluctuating Complex Quantity with Mean 1 (0) for the Real (Imaginary) Part. Predicted by RMT

\[ Z_{avg} = Z_R + R_R \sum_{b(l)} p_{b(l)} \sqrt{D_{b(l)}} e^{ik(l+L_{Port})-i\pi/4} \]

Complex Radiation Impedance (characterizes the non-universal coupling)

Index of ‘Short Orbit’ of length \( l \)

Action of orbit

 Stability of orbit

Orbit Stability Factor:
- Segment length
- Angle of incidence
- Radius of curvature of wall
Assumes foci and caustics are absent!

Orbit Action:
- Segment length
- Wavenumber
- Number of Wall Bounces
Perfectly absorbing boundary

\[ p_{b(l)} \] survival probability of ‘short orbit’ \( b(l) \) in ensemble

The waves do not return to the port

The Extended Random Coupling Model  
Non-Universal Wave Scattering Properties

We use the extended Random Coupling Model to uncover the universal impedance (reaction) matrix:

\[ z_n = R_{avg}^{-1/2} \left( Z - iX_{avg} \right) R_{avg}^{-1/2} \]

In practice we determine \( Z_{avg} \) in one of two ways:

\[ Z_{avg} = \langle Z \rangle \]

Ensemble average of cavity \( Z \)

Or by combining \( Z_R \) with ‘short orbit’ calculations:

\[ Z_{orbit} = iX_R + R_R^{1/2} \left( \sum_{m=1}^{M} \zeta_m \right) R_R^{1/2} \]

Once \( z_n \) is found, we calculate \( s_n \) as

\[ s_n = \left( z_n - 1 \right) / \left( z_n + 1 \right) \]
Comparison of Fading Amplitudes

Rayleigh distribution ($\sigma = 0.226$)

RMT model ($\alpha = 0.5$)

Experiment data (6.0 - 8.0 GHz)

Difference expected since we are not in the limit $\alpha \to \infty$

$\sigma \simeq (8\pi \alpha)^{-0.5} = 0.282$. 
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Predictions of Fading Amplitudes Including System-Specific Features

\[ Z^{(\text{theo})} = iX_{\text{avg}} + R_{\text{avg}}^{1/2} \left( z_n^{(\text{theo})} \right) R_{\text{avg}}^{1/2} \]

- Predicted Raw Impedance
- System-specific details
- RMT

This can predict fading amplitudes described by the Rician model:

\[ P(x; \sigma, \nu) = \frac{x}{\sigma^2} e^{-\left(x^2 + \nu^2\right)/2\sigma^2} I_0 \left( \frac{\sqrt{\nu}x}{\sigma} \right) \]

**Rician fading**

(two parameters)

**Experimental data**

(10.0~10.5 GHz)

Rician distribution

(\(\sigma = 0.20\), \(\nu = 0.11\))
Extended RCM Prediction of Rician Fading Parameters

\[ P(x; \sigma, \nu) = \frac{x}{\sigma^2} e^{-\left(x^2 + \nu^2\right)/2\sigma^2} I_0 \left( \frac{x\nu}{\sigma^2} \right) \]

\( S_{avg} \) is from \( Z_{avg} \) for 100 realizations of the billiard, \(|<S>_{21}|\)

\( S_{orbit} \) is from \( Z_{orbit} \) and includes \( Z_R \) and all short orbits up to 200 cm length (~ 6 \( \ell \))

\( \nu \) is from a best fit of data in a 2 GHz window to the Rician distribution

\( \sigma \) is from Rayleigh fits to \( s_n \)
Fading in the Low-Loss Limit

Prediction: Rayleigh and Rician models should fail in the low-loss limit

Superconducting Microwave Billiard
From A. Richter, Darmstadt

Conclusions

The extended Random Coupling correctly predicts Rayleigh and Rician fading in the high-loss ($\alpha >> 1$) limit

The fading parameters now have physical interpretations:
- Rayleigh $\sigma^2 \sim 1/\alpha$
- Rician $\nu$ includes antenna properties, direct signal, short orbits

In the low-loss limit ($\alpha << 1$) the Rayleigh and Rician models fail to describe the data
The RCM-base model works in this limit

Thank You!
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