Shift of resonance widths as a probe of non-orthogonality

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in collaboration with

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Motivation & outline

- Perturbation theory as a powerful tool to

  ... extend the class of explorable systems

  ... probe them experimentally (scattering setup)

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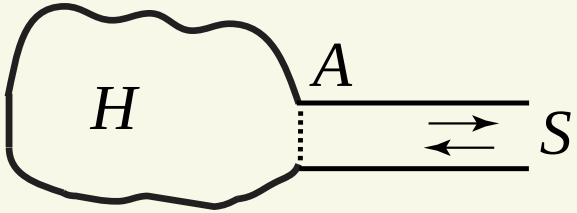
▷ How do resonance states change under perturbations?

▷ What is their parametric statistics in chaotic regime?

Resonances in scattering

N levels (internal Hamiltonian $H = H^\dagger$) coupled to M channels via A_n^c



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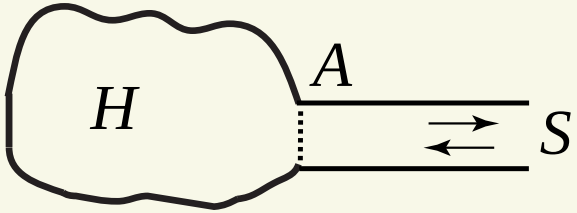

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- Scattering matrix $= \frac{\text{outgoing amplitude}}{\text{incoming amplitude}}$: $(\dim S = M)$

$$S_{\text{res}}(E) = \frac{1 - iK(E)}{1 + iK(E)}\,, \quad \text{with reaction matrix } K(E) = \tfrac{1}{2} A^\dagger \tfrac{1}{E-H} A$$

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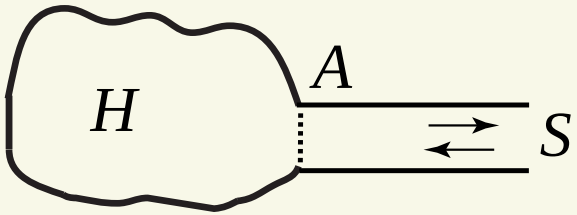

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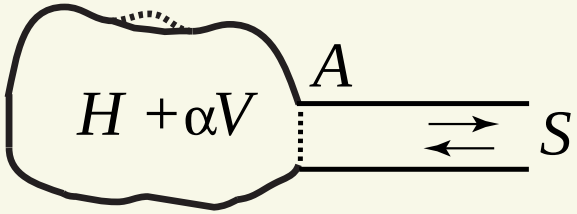
- Equivalent representation Mahaux, Weidenmüller (1969); Livšic (1973)

$$S_{\rm res}(E) = 1 - iA^\dagger \frac{1}{E - \mathcal{H}_{\rm eff}} A$$

in terms of the effective non-Hermitian Hamiltonian: $(\dim \mathcal{H}_{\rm eff} = N)$

$$\mathcal{H}_{\rm eff} = H - \tfrac{i}{2} A A^\dagger\,, \quad \text{resonances} \rightsquigarrow \text{complex e.v. } \mathcal{E}_n = E_n - \tfrac{i}{2}\Gamma_n$$

Perturbing the internal region



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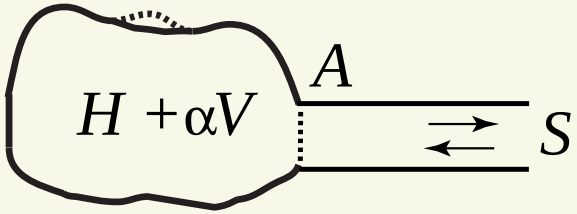

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  $\langle L_n|R_m\rangle = \delta_{nm}$ but $\langle L_n| \neq (|R_n\rangle)^\dagger \rightsquigarrow$ nonorthogonality matrix U_{nm}

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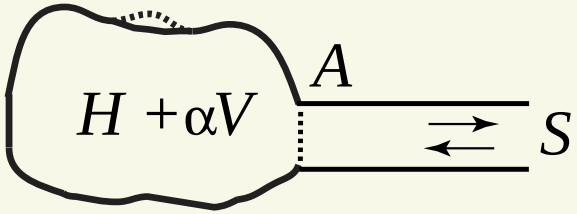

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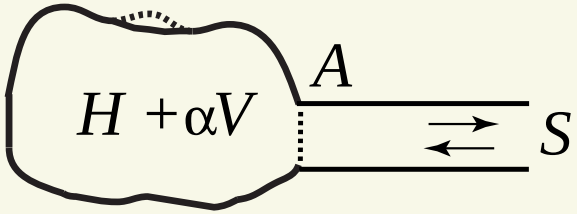

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- Shift of the resonance width $\delta\Gamma_n$ reads

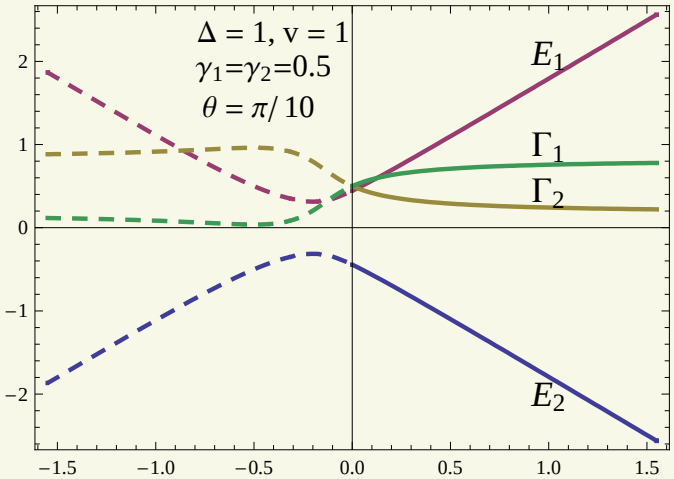
$$\delta\Gamma_n = i\alpha\sum_m (U_{nm}V_{mn} - V_{nm}U_{mn}), \quad \text{where } V_{nm} = \langle R_n|V|R_m\rangle$$

  $\rightsquigarrow$ nonzero $\delta\Gamma_n$ is an indicator of nonorthogonality

An example: Two-level system

Level separation Δ, coupling $(\hat{A}\hat{A}^\dagger)_{12} = \sqrt{\gamma_1\gamma_2}\cos\theta$, with $\gamma_{1,2} = ||A_{1,2}||^2$

$$\frac{1}{2}\begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix} - \frac{i}{2}\begin{pmatrix} \gamma_1 & \sqrt{\gamma_1\gamma_2}\cos\theta \\ \sqrt{\gamma_1\gamma_2}\cos\theta & \gamma_2 \end{pmatrix} + \alpha\begin{pmatrix} 1 & v \\ v & -1 \end{pmatrix}$$



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$\Gamma_1 + \Gamma_2 = \text{const}$
(for any $V = V^\dagger$)

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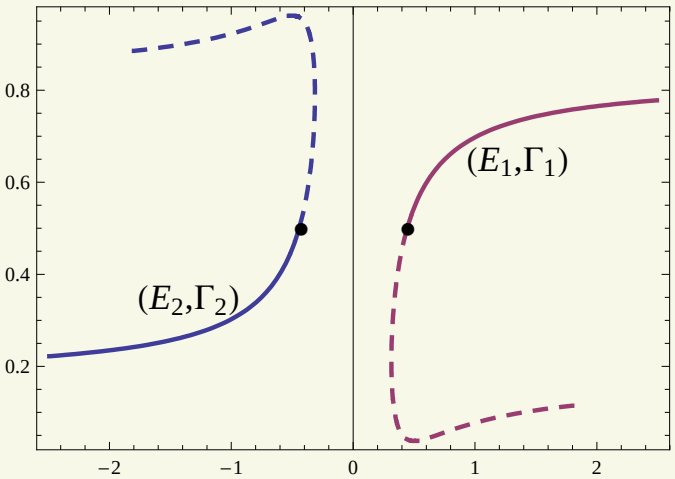

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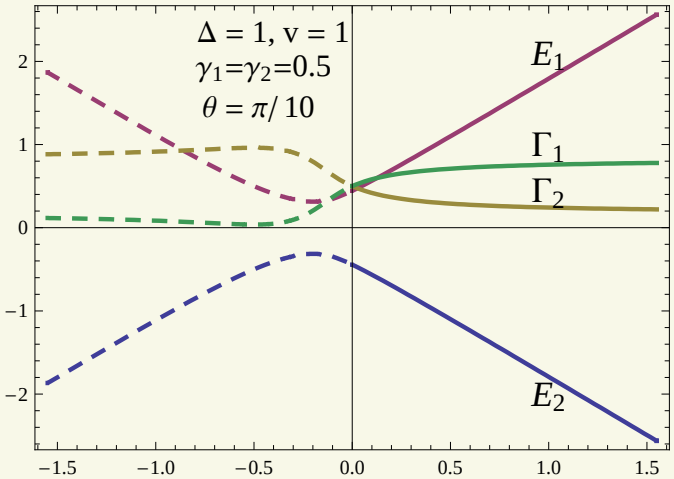

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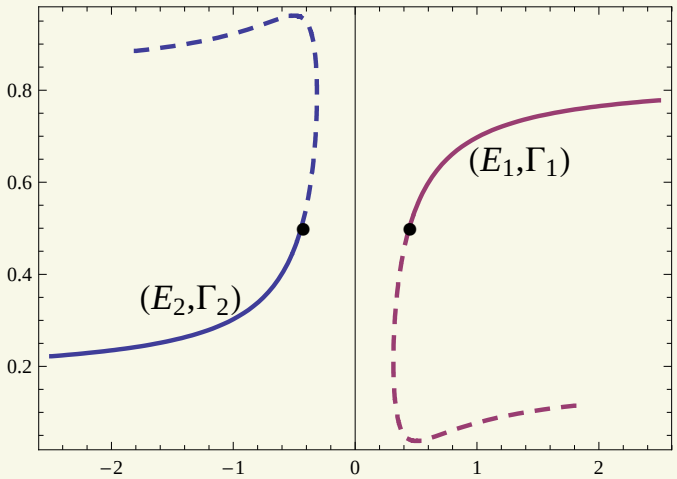
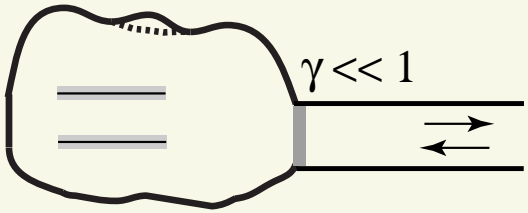


- Nonorthogonality matrix $U = [(1+|f|^2)1_2 + 2\mathrm{Re}(f)\,\sigma_y]/|1-f^2|^2$ with complex $f = \frac{\sqrt{\gamma_1\gamma_2}\cos\theta}{\delta+\sqrt{\delta^2-\gamma_1\gamma_2\cos^2\theta}}$, where $\delta = \Delta - \frac{i}{2}(\gamma_1-\gamma_2)$
- Width 'velocities' are due to the off-diagonal element of U:

$$\delta\dot{\Gamma}_1 = 4v\mathrm{Re}(f)/|1-f^2|^2 = -\delta\dot{\Gamma}_2$$

Weakly open systems



$\mathcal{H}_{\text{eff}} = H - \frac{i}{2}\gamma \hat{A}\hat{A}^{\dagger}$, with $\gamma \ll 1$

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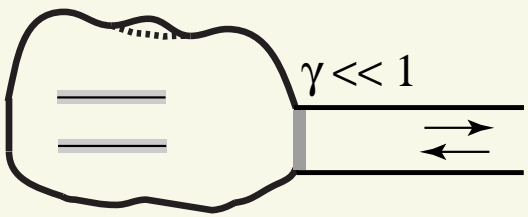

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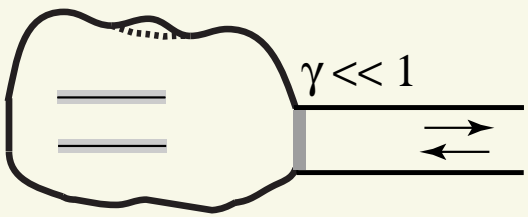

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- Result of resonance interference (interaction via common channels)

... governed by rank-two operator G_n

... different from other non-orthogonality measures:
- e.g. Peterman's factor U_{nn} (Schomerus et al, 2000)
- or complexness factor $(\mathrm{Im}\psi_n)^2/(\mathrm{Re}\psi_n)^2$ (DS, Legrand, Mortessagne, 2006)

Chaotic cavities

- **Statistical approach:** replace H with a random operator

  Wigner, Dyson ('60s), Bohigas, Giannoni, Schmidt (1984)

  H taken from appropriate ensemble of random matrices $\leftrightsquigarrow$ **RMT**
  $+$ symmetry constraints on H (e.g. $H^T = H$ for time-reversal)

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$$\langle(\cdots)\rangle = \text{const} \int (\cdots) \prod_{n\neq m} |E_n - E_m|^\beta \prod_n e^{-\frac{N\beta}{4}E_n^2} dE_n$$

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$$\kappa_n = \frac{\pi\Gamma_n}{\gamma\Delta}: \quad \mathrm{Prob}(\kappa) \propto \kappa^{M\beta/2-1} e^{-\beta\kappa/2}$$

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▷ What are the statistical properties of width shifts $\delta\Gamma_n$?

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- Geometrical representation in terms of scalar products of M-vectors with lengths κ_n and projections $z_m^{(n)}$ Poli, DS, Legrand, Mortessagne (2009)
- Rescaling in natural units (to get rid of non-universal features)

$$y_n = \frac{N\delta\Gamma_n}{2\alpha\gamma\sqrt{\mathrm{Tr}(V^2)}} = \frac{\sqrt{\kappa_n}}{\pi}\sum_{m\neq n}\frac{\Delta\mathrm{Re}(z_m^{(n)*}v_m^{(n)})}{E_n - E_m}$$

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$$\mathcal{P}_M(y) = \Delta\langle\sum\delta(E_n)\overline{\delta(y-y_n)}\rangle$$

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- Reduction to the spectral determinant problem for F.T.

$$\overline{e^{-i\omega y_n}} = \prod_{m\neq n}\frac{|E_n-E_m|^\beta}{[(E_n-E_m)^2+\kappa_n(\omega\Delta/\pi\sqrt{\beta})^2]^{\beta/2}}$$

$$\rightsquigarrow \mathrm{const}\left\langle\frac{\det(H_1)^{2\beta}}{\det[H_1^2+(\omega\Delta/\pi\sqrt{\beta})^2]^{\beta/2}}\right\rangle_{N-1} \equiv C_{N-1}^{(\beta)}(\omega)$$

Exact result and discussion

$$\mathcal{P}_M(y) = \int_0^\infty \frac{d\kappa}{\sqrt{\kappa}}\, \chi^2_{M\beta}(\kappa)\, \phi^{(\beta)}\left(\frac{y}{\sqrt{\kappa}}\right), \qquad \text{with} \begin{cases} \phi^{(1)}(y) = \frac{4+y^2}{6(1+y^2)^{5/2}} \\ \phi^{(2)}(y) = \frac{35+14y^2+3y^4}{12\pi(1+y^2)^4} \end{cases}$$

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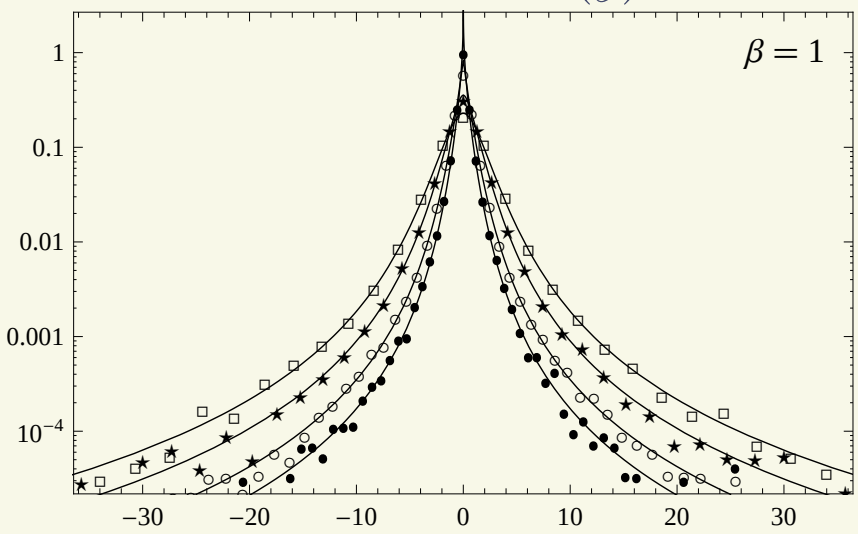
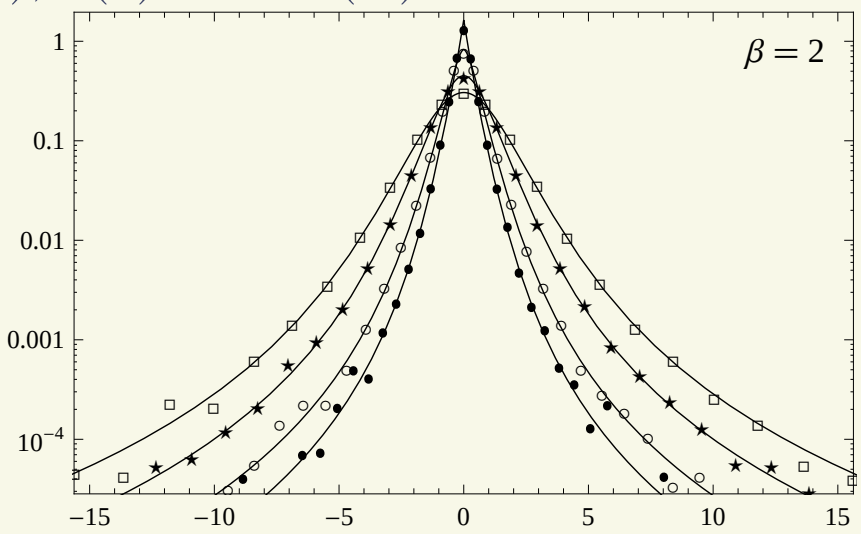
Distribution $\mathcal{P}_M(y)$ at $M = 1\,(\bullet), 2\,(\circ), 5\,(\star)$ and $10\,(\square)$ channels:



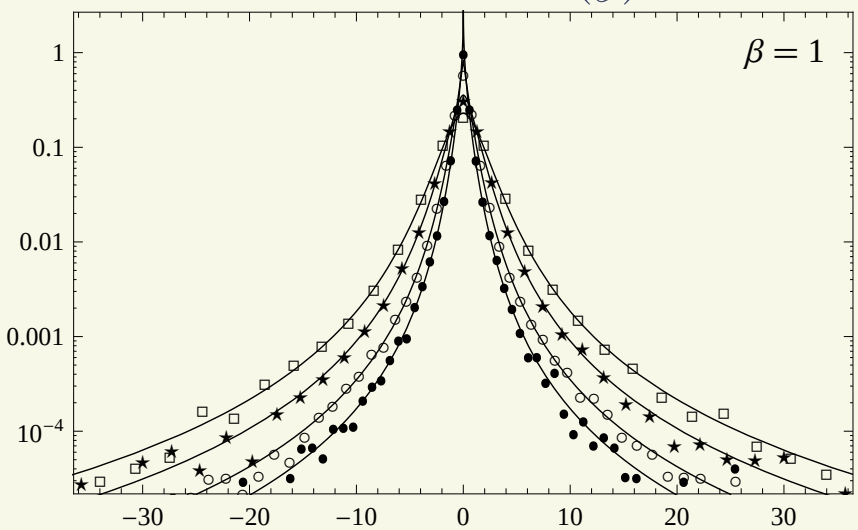


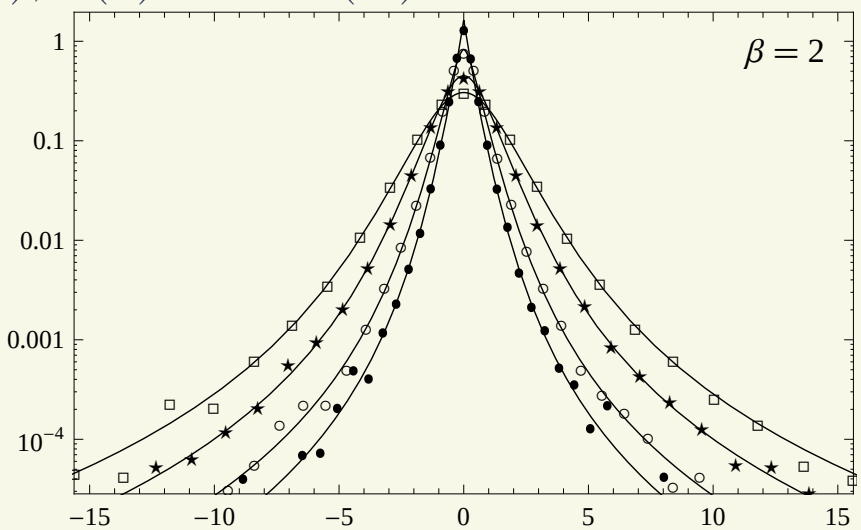
Exact result and discussion

$$\mathcal{P}_M(y) = \int_0^\infty \frac{d\kappa}{\sqrt{\kappa}}\, \chi^2_{M\beta}(\kappa)\, \phi^{(\beta)}\Big(\frac{y}{\sqrt{\kappa}}\Big), \qquad \text{with} \begin{cases} \phi^{(1)}(y) = \frac{4+y^2}{6(1+y^2)^{5/2}} \\ \phi^{(2)}(y) = \frac{35+14y^2+3y^4}{12\pi(1+y^2)^4} \end{cases}$$

where $\phi^{(\beta)}(y)$ is the F.T. of spectral determinant $C_\infty^{(\beta)}(\omega)$

Distribution $\mathcal{P}_M(y)$ at $M = 1\,(\bullet), 2\,(\circ), 5\,(\star)$ and $10\,(\square)$ channels:





- Far tails $\propto y^{-(\beta+2)}$ $\leftrightsquigarrow$ level repulsion $\propto |E_{n+1} - E_n|^\beta$
- At $M \gg 1$ converges to $\phi^{(\beta)}(\frac{y}{\sqrt{M}})$ $\rightsquigarrow$ a way to measure $C_\infty^{(\beta)}(\omega)$?!
- $\mathrm{var}(\Gamma)$ as a control parameter for nonorthogonality

Conclusion & outlook

- Parametric motion of resonance states in open systems
- Access to spatial properties via purely spectral tools
- Shift of resonance widths as a signature of nonorthogonality
- Distribution of width shifts in weakly open chaotic systems

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- Further study:
  - ▷ Global vs local perturbations
  - ▷ Generalisation to arbitrary modal overlap
  - ▷ Correlation properties, and other statistics
  - ▷ "Making sense" of pseudospectra (e.g. $U_{nn} =$ condition number)