Spontaneous emission in the presence of a dielectric cylinder

Władysław Żakowicz and Maciej Janowicz
Institute of Physics, Polish Academy of Sciences, Aleja Lotników 32/46, Warsaw 02-668, Poland
(Received 24 March 1999; revised manuscript received 27 January 2000; published 16 June 2000)

Spontaneous emission of photons by an atom placed near a dielectric cylindrical waveguide at an arbitrary position is analyzed using global free electromagnetic modes, satisfying necessary continuity conditions at the slab boundaries. These modes include the modes extended over the whole space (travelling) and the modes trapped to a waveguide (waveguided). To describe angular properties of the spontaneous emission, a parametrization of the travelling modes by the outgoing waves has been proposed. Angular characteristics of the travelling photon emission, distributions of the trapped photon emission, as well as the global decay rates have been calculated using the quantum approach and the standard perturbation theory. Difficulties due to the presence of sharp resonances (whispering gallery modes) among the travelling photons are pointed out.

PACS number(s): 42.50.Ct, 32.80.–t, 12.20.Ds

I. INTRODUCTION

Nowadays, it is well understood and experimentally verified that various devices and material objects, e.g., resonators, waveguides, dielectric, and conducting interfaces influencing electromagnetic fields modify the atomic radiative processes determining decay times, resonant line shifts, and emission spectral line shapes (see, e.g., [1]).

These effects have been intensively studied for the last two decades within the new branch of physics of the photon-atom interactions called cavity quantum electrodynamics (CQED). Most theoretical analyses in CQED effects either use very simple models of “cavities” (very often one dimensional) or, when keeping a more realistic structure of the electromagnetic field, restrict themselves to very simple environments of an atom: planar dielectric and conducting interfaces, and planar waveguides (cf. [1]).

The purpose of the present paper is to analyze both the spontaneous emission rates and the radiation pattern of an atom radiating near a cylindrical dielectric waveguide. To obtain these results we make use of the mode decomposition of the electromagnetic field. In a recent paper [2] the mode-decomposition approach has been judged as most valuable but very complicated and difficult (so difficult that it has never been used in the case of cylindrical geometry). Here, most of the difficulties have been successfully overcome and results relevant to possible experiments have been obtained. Thus, the present work appears as a natural follow up to our papers [3] and [4] dealing with the macroscopic quantum electrodynamics in planar geometries from the point of view of the mode decomposition.

Cylindrical systems have recently been investigated in [2] and [5]. While the two papers differ in theoretical methods used, they both assume that the atom is located on the axis of the cylinder; this assumption allows significant simplification as well as drastic approximations of the treatment. The present paper does not make such assumption about the position of the atom and takes into account all of the modes of the electromagnetic field.

Generally, in the presence of a dielectric cylinder one can distinguish two types of electromagnetic waves. The first type includes the waves trapped by the waveguide which are confined to its interior and proximity, and able to propagate only along its length. The second class corresponds to the waves, which at a long distances from the cylinder, are like free waves and are modified due to the scattering processes.

Our work belongs to a well-defined stream of recent contributions to the perturbative CQED in which the spontaneous emission rates and radiation patterns near dielectric interfaces are obtained for various geometrical arrangements. The rate and pattern for an atom near a single vacuum-dielectric interface using the mode-decomposition approach have been obtained in [3]. The spontaneous emission problem for the planar waveguide has been solved in [6] and [4]; the latter paper also contains an analysis of the radiation pattern. Experiments on the spontaneous emission in the presence of the dielectric interface have been reported by Snoeks and co-workers [7]. The authors of [8] independently derived the results contained in [4], generalized the approach to the case of the planar waveguide kept between two different media, and made a comparison with experimental findings. For single- and multilayer structures, both the rate and emission pattern have been discussed in [9]. In this paper an indirect method employing the Lorenz reciprocity theorem has been used to obtain the spatial distribution of radiation. This approach has provided the same radiation patterns in the case of a single interface as those obtained in [3] by using mode decomposition of the field.

We have shown in [3] and [4] that the properties of the angular distribution of emitted radiation can be obtained in the easiest way when the quantum electromagnetic field operators are expressed in terms of outgoing modes and outgoing creation and annihilation operators. These outgoing modes are completely equivalent to the usually employed incoming modes based on incoming (or incident) waves. The use of incoming modes in discussion of angular emission properties requires a very thorough treatment of interference effects as the comment [10] to the paper [11] pointed out.

The rest of the paper is organized as follows. Section II is devoted to the derivation and illustration of the properties of the classical modes of a dielectric cylinder system, both in the waveguiding and scattering regimes. The trapped (or waveguiding) modes and two different kinds of traveling modes (the ones based on cylindrical waves and those based
on plane waves) are described. This section also contains a short discussion of a distinguished type of traveling modes called the whispering gallery modes.

Section III describes quantization of the electromagnetic field in the presence of a dielectric cylinder. The excitations of the field modes are identified with photons while the modes themselves represent the wave functions of the photons. This procedure and, in particular, its application in the discussion of the spontaneous emission follows the similar discussion for planar systems given in [3] and [4], while the general rules and forms of the quantization itself are inspired by [12].

In Sec. IV we provide basic formulas to calculate various characteristics of spontaneous emission including its rate and angular properties of emitted radiation.

In Sec. V we provide the detailed discussion of the angular properties of radiation emitted spontaneously by the atom. The discussion is based on the orthodox perturbation theory [13,14] modified due to the particular atom environment. This approach leads to the Fermi golden rule modified according the photon wave functions. A possible weakness of this approach due to the presence of the whispering gallery modes is pointed out. Their existence for all frequencies causes exceptionally strong atom-field coupling in an extremely narrow range of the photon parameters. Section V is also devoted to the presentation of the spontaneous emission rate as a function of the polarization of the atomic dipole and of the distance of the atom from the axis of the cylinder. Contributions of various types of modes to the decay rate are given. Section VI contains the summary and several final remarks.

The Appendix discusses the orthogonality properties of traveling modes.

II. FIELD MODES

We study the spontaneous emission of an atom placed near or inside a dielectric cylindrical waveguide of diameter $a$, made of ideal, non-dispersive, and lossless dielectric characterized by a dielectric permittivity $\varepsilon_d$. The waveguide is placed along the $z$ axis. To describe the spontaneous emission we have to define quantum states of the electromagnetic field identified with photon states.

The description of the quantum electromagnetic field starts with classical solutions of the homogeneous Maxwell equations that depend on time as $\exp(-i\omega t)$ and satisfy proper continuity conditions on the dielectric boundaries. These harmonic, monochromatic fundamental solutions form a set of the spatial modes, which in the framework of classical description allow us to express an arbitrary electromagnetic field and in the quantum case, after a standard quantization procedure, give an expansion of the quantum electromagnetic field operators.

The system is homogeneous in the direction along the waveguide, chosen as the $z$ axis. Hence all field components of the modes may depend on $z$ as $\exp(ik_n z)$. In such a configuration all vector components of the electromagnetic wave can be derived from the $z$ component of the electric and magnetic fields $E_z$ and $B_z$. In a homogeneous dielectric as well as the surrounding space both fields $E_z$ and $B_z$ satisfy two identical Helmholtz equations ($k_0 = \omega / c$):

$$
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \phi^2} + k_0^2 \varepsilon_r \right) \begin{pmatrix} E_z \\ B_z \end{pmatrix} = 0,
$$

where: $\varepsilon = \varepsilon_d > 1$ for $r < a$, and $\varepsilon = \varepsilon_r = 1$ for $r > a$.

Although the equations for $E_z$ and $B_z$ look uncoupled, they are, in fact, mutually dependent due to continuity boundary conditions for the tangent electric and magnetic fields: $E_z$, $E_\phi$, $B_z$, and $B_\phi$ at $r = a$.

The remaining field components can be easily derived from the Maxwell equations:

$$
E_r = \frac{1}{k_0 \varepsilon} \frac{\partial E_z}{\partial r} + \frac{i \varepsilon}{r} \frac{\partial B_z}{\partial \phi},
$$

$$
E_\phi = \frac{1}{k_0 \varepsilon} \frac{i \varepsilon}{r} \frac{\partial E_z}{\partial \phi} - \frac{\partial B_z}{\partial r},
$$

$$
B_r = \frac{1}{k_0 \varepsilon} \left( - \frac{i \varepsilon}{r} \frac{\partial E_z}{\partial \phi} + i \frac{\partial B_z}{\partial r} \right),
$$

$$
B_\phi = \frac{1}{k_0 \varepsilon} \left( i \frac{\partial E_z}{\partial r} + \frac{\partial B_z}{\partial \phi} \right).
$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{(a) Propagation factors $\beta_n = k_n / k_0$ for all modes, ordered according to the partial wave index $n$, and the sequential index $i$. For each partial wave $n$ the values of $i$ are ordered starting from the top. (b) Inverse of the group velocities, $v^i_r$, for the above modes. The sequential values $i$ should be counted starting from the bottom.}
\end{figure}
where \( \beta = k_z l_{0}, \) and \( W^\mu = \varepsilon - \beta^2. \)

Taking into account the cylindrical symmetry of the system one may look for solutions of the Maxwell’s equations in the terms of cylindrical radial waves:

\[
\begin{bmatrix}
E^w_z(r, \phi, z) \\
B^w_r(r, \phi, z)
\end{bmatrix} = \frac{e^{ik_z z} e^{i\phi}}{\sqrt{2\pi} \sqrt{2\pi}} \begin{bmatrix}
E_{z, k_z, n}(r) \\
B_{z, k_z, n}(r)
\end{bmatrix},
\] (6)

and Eqs. (1) reduce to a Bessel equation for \( k^2 < \varepsilon k_0^2, \) and modified Bessel one for \( k^2 \geq \varepsilon k_0^2 \) (which can happen for \( \varepsilon = \varepsilon_0 = 1). \)

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + k^2 - \frac{n^2}{r^2} \right) \begin{bmatrix}
E_{z, k_z, n}(r) \\
B_{z, k_z, n}(r)
\end{bmatrix} = 0.
\] (7)

Here, \( n \) can take any positive or negative integer value and \(-\infty < k_z < \infty.\)

### A. Trapped (guided) modes

The waves satisfying the condition \( \varepsilon_z k_0^2 < k_z^2 < \varepsilon_0 k_0^2 \) are attached to the waveguide.

\[
M^w_n = \begin{bmatrix}
J_n(\rho_n) & 0 & -K_n(\rho_n) & 0 \\
0 & J_\nu(\rho_n) & 0 & -K_\nu(\rho_n) \\
-\frac{n\beta}{q_d^2 k_0^2} J'_n(\rho_n) + \frac{1}{q_d} J_n(\rho_n) & -\frac{n\beta}{q_w^2 k_0^2} K'_n(\rho_n) + \frac{1}{q_w} K_n(\rho_n) \\
-\frac{n\varepsilon_0}{q_d^2 k_0^2} J'_\nu(\rho_n) + \frac{1}{q_d} J_\nu(\rho_n) & -\frac{n\varepsilon_0}{q_w^2 k_0^2} K'_\nu(\rho_n) + \frac{1}{q_w} K_\nu(\rho_n)
\end{bmatrix},
\] (11)

where \( \rho_n = k_0 q_d a, \) and \( \rho_\nu = k_0 q_w a. \) With the imaginary unit introduced to magnetic terms in Eqs. (8)–(9) both the coefficients entering \( X_n \) and the matrix \( M^w_n \) are purely real.

The nonvanishing solutions of this set of equations exist when

\[
\det M^w_n(\beta) = -\frac{1}{q_d^2 q_w} \left[ q_d J_n(\rho_n) K'_n(\rho_n) - q_w J'_n(\rho_n) K_\nu(\rho_n) \right] \times \left[ \varepsilon_0 q_d J_\nu(\rho_n) K'_\nu(\rho_n) - \varepsilon_0 q_w J'_\nu(\rho_n) K_n(\rho_n) \right] + \beta^2 \frac{n^2}{a^2 k_0^2} \left[ \frac{1}{q_d^2} - \frac{1}{q_w^2} \right]^2 J_n(\rho_n)^2 K_n(\rho_n)^2 = 0,
\] (12)

representing a characteristic equation for the wave propagation problem, with a discrete set of solutions for the propagation wave vectors \( k_{zni} = k_0 \beta_{ni}. \)

Inside the cylinder the solutions of Eq. (7) regular at \( r = 0 \) can be looked for as

\[
\begin{bmatrix}
E_{z, k_z, n}(r) \\
B_{z, k_z, n}(r)
\end{bmatrix} = \begin{bmatrix}
a_n \\
b_n
\end{bmatrix} J_n(k_0 q_d r), \quad r < a,
\] (8)

while in the external region the solutions vanishing at infinity can be assumed as

\[
\begin{bmatrix}
E_{z, k_z, n}(r) \\
B_{z, k_z, n}(r)
\end{bmatrix} = \begin{bmatrix}
c_n \\
d_n
\end{bmatrix} K_n(k_0 q_w r), \quad r > a
\] (9)

where \( q_d = \sqrt{\varepsilon_d - \beta^2}, \) and \( q_w = \sqrt{\varepsilon_w - \beta^2} \) and \( J_n \) are the ordinary Bessel functions and \( K_n \) are the modified Bessel function of the \( n \)th order.

The boundary continuity conditions imposed on the field components \( E_z, E_\phi, B_z, \) and \( B_\phi \) at the waveguide surface lead to a set of linear equations \( (X_n = \{a_n, b_n, c_n, d_n\}), \)

\[
M^w_n \cdot X_n = 0,
\] (10)

with a matrix \( M^w_n \) given by

Only in the case \( n = 0 \) the corresponding modes can be classified as transverse magnetic (TM) or transverse electric (TE). For nonzero \( n \) the above is not possible and all the coefficients \( \{a_n, b_n, c_n, d_n\} \) are different from zero.

The solutions of the characteristic equations, considered either for fixed waveguide radius \( a \) as a function of the frequency \( \omega, \) or for fixed frequency \( \omega \) as a function of the cylinder radius \( a \) are often discussed in many textbooks on waveguides [15,16]. For a given \( n, \) the modes will be counted and labeled by a sequential parameter \( i. \)

We point out here that for the waveguides admitting many modes, e.g., when \( \lambda \ll a, \) the propagation constants \( \beta_{ni}, \) characterizing the phase velocities of the modes, \( v_{ph} = c/\beta_{ni}, \) form a regular pattern on a plane parametrized by the azimuthal, or partial wave indices \( n, \beta. \) An example is shown in Fig. 1(a). It will be shown later that a rate of the spontaneous emission depends on the group velocity of the mode, \( v_{grp} = \partial \omega / \partial k_z. \) The corresponding parameters...
$v_r^{-1} = \beta_n + \omega \partial \beta_n / \partial \omega$ also form a very regular pattern, shown in Fig. 1(b). The number of modes for each $n$, denoted as $\nu(n)$, is a decreasing function of $|n|$. Beyond certain $n = n_{max}$ the waveguide considered has no more trapped discrete modes. Notice, however, that very thin waveguides still permit single solutions for $n = \pm 1$ with $\beta = \pm \beta_1$, while all other modes, including $n = 0$ modes, are below their thresholds.

When $\det M_n = 0$ we can assume that one of the coefficients $\{a_n, b_n, c_n, d_n\}$ is arbitrary and find the remaining ones as functions of this selected coefficient. Knowing $\{a_n, b_n, c_n, d_n\}$ and, therefore, $\{E, B\}$ given by Eqs. (8)–(9) we find the remaining field components $E_r, E_\phi, B_r, B_\phi$ with the help of Eqs. (2)–(5). Imposing the normalization condition

$$\int_0^\infty dr r [e^{E_w(s,r)} \cdot E_{n,i}(r) + e^{-E_w(s,r)} \cdot B_{n,i}(r)] = 1.$$  

(13)

fixes the selected parameter and uniquely determines the electromagnetic field of a particular mode. This procedure is straightforward but rather tedious when treated analytically and for our application to the spontaneous photon emission problem the waveguided modes will be handled numerically.

The numbers of modes, including different partial waves, are shown in Fig. 2 for the selected values of the $a/\lambda$ ratio. We are going to treat the above guided wave modes as the bases for field quantization. It is therefore more convenient to treat the mode parameters $k_z$, $n$, and $c$ as the independent parameters while the frequency of the mode $\omega$ becomes a function of these independent variables, i.e., $\omega = \omega(k_z)$. This expression indicates that $\omega$ can be represented by a multivalued function of the longitudinal wave vector $k_z$ and the different branches are labeled by $n$ and $c$. Figure 3 illustrates these functions, shown separately for each partial mode. The different trapped or waveguided modes, defined as

$$\begin{align*}
E_{k_z,n,i}(r) &= e^{i k_z r} e^{i \omega \phi} \\
B_{k_z,n,i}(r) &= -\frac{2 \pi}{e^{i k_z r}} B_{k_z,n,i}(r)
\end{align*}$$

(14)

are mutually orthogonal and are normalized according to

$$\int d_3r \left( e^{E_w(s,\cdot)} E_{k_z,n,i}(r) E_{k_z,n,i}(r) + e^{-E_w(s,\cdot)} B_{k_z,n,i}(r) B_{k_z,n,i}(r) \right) = \delta(k_z-k_z').$$

(15)

B. Traveling modes

The modes satisfying the condition $|k_z| < k_0 = \omega / |\beta| < 1$ extend over the entire space and will be called traveling modes since they are not localized in the vicinity of the dielectric rod. They may be constructed from the modes of free space called the primary modes. These primary modes are modified due to the presence of the cylinder by the inclusion of secondary waves. The secondary waves are usually associated with the scattering process [30,31]. The whole traveling modes can be parametrized in the same way as free-space modes, however, one must point out that the choice of the secondary waves is still arbitrary. As the secondary waves one may take the waves propagating either outwards or towards the cylinder. In the former case the resulting complete modes, which involve both primary and secondary waves, will be called “incoming” and can be associated with the scattering process. In the latter case they will be called “outgoing” and could correspond to the problem opposite to scattering, which might be called the “assembling” problem.

Our “incoming” and “outgoing” mode functions may find analogy in the concepts of “in” and “out” states (denoted as $|\psi^+(\cdot)|$ and $|\psi^-(\cdot)|$ respectively) in the quantum scattering theory, cf., e.g. [17,18] and quantum field theory, see, e.g., [19]. There are, however, some differences. While the latter are defined through the asymptotic behavior in time and their constructions are based on retarded and advanced time-dependent propagators, our modes are connected with the stationary solutions of the field equations and are distinct through their different properties in space. Despite the conceptual difference, our “incoming” and “outgoing” modes as functions of spatial coordinates are identical with the “in” and “out” states of the scattering theory.
Two choices of primary (or free-space) modes seem to be natural. They can be based either on cylindrical waves, parametrized by \( k_z, k_z', n \), and polarizations (TE or TM) or on plane waves parametrized by the wave vector \( k \) (and polarization—either linear or circular). It is to be stressed that only the primary waves have the definite polarization, either TE for which \( E_z = 0 \), or TM for which \( B_z = 0 \). The traveling modes do not keep, in the generic case, the polarization properties of primary waves. We shall, however, apply the TE and TM terminology to the whole traveling modes according to the polarization properties of the primary fields. In the following subsections we shall consider separately these two choices of primary waves and of corresponding modes.

1. Angular partial-wave scattering modes

The solution to the Maxwell equations leading to the explicit expressions for the mode functions is based on the expansion in terms of vectorial cylindrical wave functions, which have to satisfy the continuity conditions. All these wave functions contain the common factor \( \exp(i n \phi + i k_z z) \). Their radial parts consist of the solutions to the Bessel equations and their derivatives. As a proper solution inside the cylinder the Bessel function \( J_n \) (with its derivative) must be chosen due to the regularity condition at \( r = 0 \). Outside the cylinder we have to form the radial part of the mode from two linearly independent solutions to the Bessel equations. The radial parts of primary fields, taken to be waves propagating in free space, are expressed by the Bessel functions \( J_n \). They must be supplemented by secondary fields, which contain a second independent solution to the Bessel equation that is chosen to be either the Hankel function of first kind \( H_n^{(1)} \) (representing a wave propagating outwards the cylinder) or the Hankel function of the second kind \( H_n^{(2)} \) (representing a wave propagating inwards the cylinder). Let us stress that the choice between two Hankel functions is purely conventional and equally justified as both systems can lead to the complete set of solutions. We have found that the second choice (i.e., the Hankel functions of the second kind) is more convenient in the description of spontaneous emission. All components of the electric and magnetic fields can be expressed in terms of the \( z \) components of the fields. The \( z \) components of fields emerging from the separation of variables for primary waves are given by

\[
\begin{align*}
\begin{cases}
\mathcal{E}^{v}_{z,k_z,n} \equiv E_z^0(k_0 q_v r), \\
B^{v}_{z,k_z,n} \equiv B_z^0(k_0 q_v r),
\end{cases}
\end{align*}
\]

As in the case of the waveguided modes, the calligraphic letters denote the radial parts of the field components.

The primary waves have to be supplemented by secondary waves, which can be assumed to have the following form: Inside the cylinder,

\[
\begin{align*}
\mathcal{E}^{v}_{z,k_z,n} & = A^{v}_n J_i(k_0 q_v r), \\
B^{v}_{z,k_z,n} & = B^{v}_n J_i(k_0 q_v r),
\end{align*}
\]

and outside the cylinder,

\[
\begin{align*}
\mathcal{E}^{v}_{z,k_z,n} & = C^{v}_n H_n^{(v)}(k_0 q_v r), \\
B^{v}_{z,k_z,n} & = D^{v}_n H_n^{(v)}(k_0 q_v r),
\end{align*}
\]

where \( q_v = \sqrt{\varepsilon_v - k_0^2} \) and \( q_v = \sqrt{\varepsilon_v - k_0^2} = k_0 / k_0 \) and \( J_n \) are the ordinary Bessel functions and \( H_n^{(v)} \) are the Hankel function of the \( n \)th order and \( v \)th kind. The case \( v = 1 \) corresponds to the scattering case while \( v = 2 \) corresponds to the "assembling" case.

Similarly as for the waveguided modes, the coefficients \( A^{v}_n, B^{v}_n, C^{v}_n, D^{v}_n \) are determined from the continuity conditions imposed on the tangential components of the electric and magnetic field. Here, however, we have to solve the inhomogeneous system off linear equations:

\[
\mathbf{M}_n^{v} Y_n^{v} = P_n,
\]

where

\[
Y_n^{v} = \{A^{v}_n, B^{v}_n, C^{v}_n, D^{v}_n\},
\]

\[
P_n = \{\mathcal{E}^{F}_{z,k_z,n}, B^{F}_{z,k_z,n}, C^{F}_{\phi,k_z,n}, B^{F}_{\phi,k_z,n}\}
\]

with

\[
E_{z,k_z,n}^{F} = E_z^0 J_n(k_0 q_v a),
\]

\[
B_{z,k_z,n}^{F} = B_z^0 J_n(k_0 q_v a),
\]

\[
E_{\phi,k_z,n}^{F} = \frac{1}{q_v} \left( -\frac{\beta n}{k_0 a} J_n(k_0 q_v a) E_z^0 - i q_v J'_n(k_0 q_v a) B_z^0 \right),
\]

\[
B_{\phi,k_z,n}^{F} = \frac{1}{q_v} \left( i q_v J_n(k_0 q_v a) E_z^0 - \frac{\beta n}{k_0 a} J_n(k_0 q_v a) B_z^0 \right).
\]
The explicit solutions for the coefficients \( \{A_n^p, B_n^p, C_n^p, D_n^p\} \) are given by

\[
A_n^p W^p = q_d^2 q_v \{ H_n^{(p)}(\rho_0) J_n(\rho_d) - H_n^{(p)}(\rho_c) J_n(\rho_d) \} - q_v H_n^{(p)}(\rho_c) J_n(\rho_d) - i \beta \eta B_n^{2q} H_n^{(p)}(\rho_c) \times \{ \epsilon_v q_d J_n(\rho_d) \}
\]

\[
B_n^p W^p = q_d^2 q_v \{ H_n^{(p)}(\rho_0) J_n(\rho_d) - H_n^{(p)}(\rho_c) J_n(\rho_d) \} - \epsilon_v q_v H_n^{(p)}(\rho_c) J_n(\rho_d) + i \beta \eta E_v^0 H_n^{(p)}(\rho_c) J_n(\rho_d) \times J_n(\rho_d) \}
\]

\[
C_n^p W^p = -E_v^0 q_d^2 \{ H_n^{(p)}(\rho_0) J_n(\rho_d) - H_n^{(p)}(\rho_c) J_n(\rho_d) \} - \epsilon_v q_d J_n(\rho_d) J_n(\rho_d) - \epsilon_v q_v J_n(\rho_d) J_n(\rho_d) \}
\]

\[
D_n^p W^p = -B_n^{2q} q_v J_n(\rho_d) \}
\]

\[
W^p = q_d^2 q_v \{ q_d H_n^{(p)}(\rho_c) J_n(\rho_d) - q_v H_n^{(p)}(\rho_c) J_n(\rho_d) \}
\]

\[
\times \{ \epsilon_v q_d H_n^{(p)}(\rho_c) J_n(\rho_d) - \epsilon_v q_v H_n^{(p)}(\rho_c) J_n(\rho_d) \}
\]

\[
- \beta^2 \eta^2 H_n^{(p)}(\rho_c)^2 J_n(\rho_d)^2 (q_d^2 - q_v^2)^2.
\]

where \( \rho_c = k_0 q_d a, \rho_d = k_0 q_d a, \) and \( \eta = n/k_0 a. \)

As a test of the validity of these complicated formulas, the following unicity condition has been checked:

\[
E_z^{02} + B_z^{02} = 2 C_n + E_z^{02}^2 + |2 D_n + B_z^{02}|^2.
\]

Testing our numerical programs we have found this condition to be satisfied with the accuracy better than \( 10^{-15}. \)

One can distinguish two independent sets of modes corresponding to two different field polarizations. These two sets can be ascribed to two different polarizations of the primary modes. The first one, which can be named the TM polarization, has a vanishing \( B_z^f \) component (i.e., \( B_z^0 = 0 \)) while the second one, the TE polarization, has \( E_z^0 = 0 (E_z^0 = 0) \).

There exists a connection between the coefficients \( \{A_n^p, B_n^p, C_n^p, D_n^p\} \) for incoming \( (\nu = 1) \) and outgoing \( (\nu = 2) \) modes depending on the polarization of the primary wave. The transformation, which enables one to find these coefficients for \( \nu = 2 \) if one knows them for \( \nu = 1 \), is given by the following expressions:

for TM \( (\mu = 1) \) modes: \( A_1^1 = A_2^b, B_1^1 = -B_2^b, C_1^1 = C_2^b, D_1^1 = -D_2^b; \)

for TE \( (\mu = 2) \) modes: \( A_1^1 = -A_2^b, B_1^1 = B_2^b, C_1^1 = -C_2^b, D_1^1 = D_2^b. \)

Let us point out that choosing the TM (or TE) polarization in the primary wave we get, when \( n \neq 0 \) and \( k_z \neq 0 \), both coefficients \( C_n \) and \( D_n \) different from 0, i.e., the dielectric cylinder disturbs the polarization property TM (or TE) of the mode. In the general case the traveling modes, similarly as the waveguiding modes, are of a hybrid type. It is convenient to hold the polarization indices, \( \mu = 1,2 \), among the mode parameters \( n, k_z, k_\perp, k_z \), understanding that the relevant polarization property TM or TE can be attributed to the primary wave only. Although the polarization TM or TE were ascribed to the primary waves these polarizations characterize the emergent part of the modes (i.e., the outward propa-
gating part) if the outgoing modes \((\nu=2)\) are used. The genuine TM or TE polarization of the modes are spoiled by their inward propagating.

Thus, the cylindrical partial modes are given by

\[
E_{\mu,n,k_z,k_z}^c(r) = \begin{cases} 
E_{\mu,n,k_z,k_z}^d & r < a, \\
E_{\mu,n,k_z,k_z}^F + E_{\mu,n,k_z,k_z}^{(v)} & r > a,
\end{cases}
\]  

(28)

where the explicit expressions for the secondary waves read

(a) inside the dielectric \((r<a)\),

\[
E_{\mu,n,k_z,k_z}^{d}(r) = \frac{e^{ik_zr}e^{in\phi}}{\sqrt{2\pi}} \frac{1}{q^2_0} \left( i\beta q_J J'_n(k_0q_d r)A_n^\nu \\
- \frac{n}{k_0r} J_n(k_0q_d r) B_n^\nu \\
+ \epsilon_0 \frac{1}{q^2_0} \left( \frac{n\beta}{k_0r} J'_n(k_0q_d r)A_n^\nu \\
- iq_d J'_n(k_0q_d r) B_n^\nu + \epsilon e J_n(k_0q_d r) A_n^\nu \right) \right).
\]  

(29)

(b) outside the cylinder \((r>a)\),

\[
E_{\mu,n,k_z,k_z}^{v}(r) = \frac{e^{ik_zr}e^{in\phi}}{\sqrt{2\pi}} \frac{1}{q^2_0} \left( i\beta q_J H_n^{\nu'}(k_0q_d r)C_n^\nu \\
- \frac{n}{k_0r} H_n^{\nu'}(k_0q_d r) D_n^\nu \\
+ \epsilon_0 \frac{1}{q^2_0} \left( \frac{n\beta}{k_0r} H_n^{\nu'}(k_0q_d r)C_n^\nu \\
- iq_d H_n^{\nu'}(k_0q_d r) D_n^\nu + \epsilon e H_n^{\nu'}(k_0q_d r) C_n^\nu \right) \right).
\]  

(30)

And a similar expression holds for \(B_{\mu,n,k_z,k_z}^{\nu} \).

The modes defined above satisfy the following normalization and orthogonality conditions (as is shown in the Appendix):

\[
\int d^3r [\mathbf{E}_{\mu,n,k_z,k_z}^{\nu*}(r) \cdot \mathbf{E}_{\mu,n',n',k_z',k_z'}^{\nu}(r) \\
+ \mathbf{B}_{\mu,n,k_z,k_z}^{\nu*}(r) \cdot \mathbf{B}_{\mu,n',n',k_z',k_z'}^{\nu}(r)]
\]

\[
= \frac{1}{k_z} \delta_{\mu\mu'} \delta_{nn'} \delta(k_z-k_z') \delta(k_z-k_z')
\]

(31)

provided that \(E_0^\mu=k_z/\sqrt{2}k_0\), \(B_0^\mu=0\) for TM \((\mu=1)\) modes, and \(E_0^\mu=0, B_0^\mu=k_z/\sqrt{2}k_0\) for TE \((\mu=2)\).

2. Plane-wave-based modes, incident—scattered or emergent—assembled modes

An alternative—and sometimes convenient—set of electromagnetic modes is based on plane waves taken as primary fields. As before, these primary fields must be supplemented by secondary ones arising due to the presence of the cylinder. These modes can easily be described in terms of the cylindrical modes just introduced. All our previous remarks about the duality between scattered and assembling, or incoming and outgoing waves apply equally well to the case of plane-wave-based modes.

The primary plane waves are characterized by a wave vector \(\mathbf{k}\) which in the spherical coordinate system has the components:

\[
\mathbf{k} = k_0 \{ \cos \theta \cos \alpha, \cos \theta \sin \alpha, \sin \theta \},
\]

and two polarizations. In this paper we shall employ linear polarizations that can be transverse magnetic (TM, \(\mu=1\)) or transverse electric (TE, \(\mu=2\)) with respect to the \(z\) axis.

The \(z\) components of the primary waves—either \(E_z^\nu\) or \(B_z^\nu\)—are equal to

\[
N \cos \theta e^{ik_z r} e^{ik_z r} = N \cos \theta e^{ik_z r} e^{i(k_0 r - \Phi - a)}
\]

(32)

where \(k_z = k_0 \{ \cos \alpha \sin \alpha \} \) is the transverse component of the vector \(\mathbf{k}\), \(k_z = k_0 \cos \theta, \alpha \) is the polar angle in the \(z\) plane perpendicular to the \(k_z\) axis, and \(N\) is a normalization factor.

Using this expansion we can immediately write down the expression for the complete (i.e., including secondary waves) plane-wave-based modes in terms of cylindrical modes:

\[
\left\{ \mathbf{E}_{k,\mu} \right\} = N \sqrt{2}(2\pi) \sum_{n=-\infty}^{\infty} i^n e^{-in\alpha} \left\{ \mathbf{E}_{\mu,n,k_z,k_z} \right\}.
\]

(33)

Similarly as in the case of cylindrical modes, we have defined the polarization of modes according to the polarization of the plane wave (incident or emerged), which may be transverse electric (TE) or transverse magnetic (TM); however, the total electromagnetic fields of any mode are neither TM nor TE.

Taking into account orthogonality relations for the cylindrical modes, we can obtain analogous relations for the plane-wave-based modes and get the proper normalization. We find

\[
\int d^3r \left( \mathbf{e} \mathbf{E}_{k,\mu}^{\nu*} \cdot \mathbf{E}_{k,\mu} + \mathbf{B}_{k,\mu}^{\nu*} \cdot \mathbf{B}_{k,\mu} \right)
\]

\[
= N^2 2(2\pi)^2 \frac{1}{k_z} \delta(k_z-k_z') \delta(k_z-k_z') \sum_{n} e^{in(\alpha-\alpha')}.
\]

(34)
C. Resonances—whispering gallery modes

A part of all the traveling modes is to be distinguished and analyzed in some detail due to its peculiar contribution to the angular pattern of radiation emitted by the atoms near the cylinder. They are the so-called whispering gallery modes, which are of great recent interest because of their role in spherical microlaser systems.

Asymptotically all the traveling modes have similar amplitudes, but in the region near the cylinder these modes can behave very differently. The coefficients $A_n^\nu$ and $B_n^\nu$ can be very large when the determinant of $M_n$ (proportional to $W^\nu$) becomes very small. This effect can be attributed to the resonances in the system. These resonances correspond to what is called the “whispering gallery modes.” The parameters of the resonances, i.e., their widths and amplitudes, can be determined from the complex poles $\tilde{\beta}$ of the determinant of $M_n(\beta)$ extended to the complex $\beta$ plane by analytical continuation:

$$\det M_n(\tilde{\beta}) = 0,$$

where $M_n(\beta)$ has been defined in Eq. (24). We shall show below that the whispering gallery modes are, in a sense, a natural extension of the waveguiding trapped modes.

Traveling or scattering modes are labeled by continuous parameter $\beta$ satisfying $\beta^2 < 1$. Their properties analyzed in terms of partial angular modes depend on the effective dielectric step. Most of the scattered $n$th partial modes satisfying

$$n^2 \left( k^2 a^2 \right) < \beta^2 < 1,$$

remain outside the waveguide, being separated from its interior by the step dip playing the role analogous to the potential barrier in quantum mechanics (QM). A typical shape of the effective barrier determined by the dielectric constant and the centrifugal term together with some examples of the wave functions of both the waveguided modes and the resonances are displayed in Fig. 5. This figure also contains one “nonresonant” wave function for a comparison. The step dip barrier equally well isolates the waveguide interior where particular scattered modes may resemble the waveguided modes. Indeed, from Fig. 6 one may infer that the whispering gallery modes are a natural extension of the guided modes to the continuous part of the spectrum. If this step dip barrier had infinite width the waveguide would have more guided modes, with characteristic eigenvalues $\beta_{n^\nu}$ satisfying Eq. (38) in addition to those for $\beta^2 > 1$. When the width of the dip is finite the modes can penetrate the barrier and when a field is excited inside the waveguide it must decay. These characteristic roots $\beta_{n^\nu}$ of Eq. (37) are complex corresponding to resonances in the scattering. The resonant scattering waves can penetrate the forbidden barrier, forming waves inside the barrier, which fit very well, in their positions and shapes, the corresponding families of the waveguiding modes.

For dielectric cylinders and dielectric spheres these resonant waves have been known as whispering gallery modes.
The properties of resonances are illustrated in Fig. 7 which shows the propagation parameters as well as the widths and strengths represented by amplitudes of the corresponding waves inside the “barrier” of scattering resonances. This graph proves that the scattering resonances do not exist for either very small or very large values of $u$. In the first case the step dip is very shallow and the barrier is very weak and penetrable, while in the second case a bucket created by the effective dielectric step is too small to hold a bound mode, compare the case $n = 40$ in Fig. 6.

The widths of resonances dramatically decrease and their strengths increase when the potential “barrier” becomes wider. The families of resonances or whispering gallery modes are shown in Fig. 7 resemble the families of the waveguided modes previously shown in Fig. 1, except those for the guided modes. The positions and other parameters of whispering gallery mode resonances are tabulated e.g. in [20]. Most of the whispering gallery mode resonances are extremely narrow, and thus are very sensitive to the slightest losses in the system. These losses may be due to irregularities of the waveguide shape or to the internal losses of the dielectric.

III. QUANTIZATION

After getting the field modes normalized, one can follow the standard field quantization procedure and introduce creation and annihilation operators for photons in states of the corresponding modes: waveguiding, $\hat{b}_{k, n, i}$ and $\hat{b}^\dagger_{k, n, i}$, and traveling. These traveling modes can be based either on the scattered plane waves, connected with the operators $\hat{a}_{k, \mu}$ and $\hat{a}^\dagger_{k, \mu}$, or on scattered cylindrical waves, $\hat{c}_{\mu, n, k_z}$, and $\hat{c}^\dagger_{\mu, n, k_z}$.

Quantization can be based on either the incoming or outgoing mode functions and the operators $\hat{a}, \hat{b}, \hat{c}$ should carry
the index \( \nu \), which will not be written to avoid overloading of notation. Let us notice that the creation and annihilation operators corresponding to different \( \nu \) are related by a linear transformation. The following commutation relations for the creation and annihilation operators can be imposed:

1. For the guided modes,

\[
\left[ \hat{b}_{k_z, n, \ell}, \hat{b}_{k_z', n, \ell}^\dagger \right] = \delta_{n, n'} \delta_{\ell, \ell'} \delta(k_z - k_z').
\]  

(39)

2. For the traveling modes

(a) in the cylindrical partial wave representation,

\[
\left[ \hat{c}_{\mu, n, k_z, k_z', \nu}, \hat{c}_{\mu', n, r, \nu'} \right] = \delta_{\nu, \nu'} \delta_{\mu, \mu'} \delta(k_z - k_z') \delta(r - r');
\]  

(40)

(b) for the plane-wave-based traveling modes,

\[
\left[ \hat{a}_{k, \mu}, \hat{a}_{k', \mu'}^\dagger \right] = \delta_{\mu, \mu'} \delta_3(k - k'),
\]  

(41)

all other commutators vanish.

Using the defined modes and photon creation and annihilation operators, one can express the operators of the electric field in terms of the creation and annihilation operators and mode functions. The transverse part of the electric field operator is the sum

\[
\hat{E}_t = \hat{E}_w + \hat{E}',
\]  

(42)

where \( \hat{E}_w \) denotes the waveguided-mode contribution given by

\[
\hat{E}_w(r) = \frac{1}{\sqrt{\epsilon_0}} \int_{-\infty}^{\infty} dk_z N(k_z) \sum_{n=-N}^{N} \sum_{\ell=1}^{\infty} \sqrt{\omega_{n, \ell}(k_z)} \times \left[ \hat{b}_{k_z, n, \ell}, \hat{E}_{k_z, n, \ell}^w(r) \right],
\]  

(43)

while \( \hat{E}' \) denotes the traveling-mode contribution given by either

\[
\hat{E}'(r) = \frac{1}{\sqrt{\epsilon_0}} \sum_{\mu=1,2} \sum_{\nu=1,2} \int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_{k_z} \sqrt{\omega_{k_z}} \times \left[ \hat{c}_{\mu, n, k_z, k_z} \hat{E}_{k_z, n, \ell}^{\mu}(r) + \hat{a}_{k, \mu}, \hat{E}_{k_z, n, \ell}^{\mu}(r) \right],
\]  

(44)

in the plane-wave parametrization, or

\[
\hat{E}'(r) = \frac{1}{\sqrt{\epsilon_0}} \sum_{\mu=1,2} \sum_{\nu=1,2} \int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_{k_z} \sqrt{\omega_{k_z}} \times \left[ \hat{c}_{\mu, n, k_z, k_z} \hat{E}_{k_z, n, \ell}^{\mu}(r) + \hat{a}_{k, \mu}, \hat{E}_{k_z, n, \ell}^{\mu}(r) \right],
\]  

(45)

in the cylindrical-wave parametrization.

The magnetic field operators \( \hat{B}_w \) and \( \hat{B}' \) can be directly obtained from the above expressions by the replacement of the electric fields \( E \) of the modes by the corresponding magnetic \( B \) fields. The Hamiltonian of the free-electromagnetic field

\[
H_F = \frac{1}{2} \int d^3 r \left[ \hat{E}_t^2(r) \cdot \hat{E}_t(r) + \hat{B}(r) \cdot \hat{B}(r) \right]
\]  

(46)

expressed in terms of the photon creation and annihilation operators is

\[
H_F = H_F^w + H_F' = \sum_{\mu=1,2} \sum_{\nu=1,2} \int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_{k_z} \sqrt{\omega_{k_z}} \times \left[ \hat{b}_{k_z, n, \ell}, \hat{E}_{k_z, n, \ell}^w(r) + \hat{a}_{k, \mu}, \hat{E}_{k_z, n, \ell}^w(r) \right]
\]  

(47)

Or one can take \( H_F' \) in the form

\[
H_F' = \sum_{\mu=1,2} \sum_{\nu=1,2} \int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_{k_z} \sqrt{\omega_{k_z}} \times \left[ \hat{b}_{k_z, n, \ell}, \hat{E}_{k_z, n, \ell}^w(r) + \hat{a}_{k, \mu}, \hat{E}_{k_z, n, \ell}^w(r) \right]
\]  

(48)

### IV. SPONTANEOUS EMISSION AND RADIATION PATTERN

For all kinds of photons the transition probabilities of the emission, per unit time, while the dipole makes the transition between states \( \psi_f \) and \( \psi_i \) with energies \( e_f \) and \( e_i \) are

\[
P_{\alpha} = 2 \pi \frac{e^2}{\epsilon_0} |x_{fi} \cdot E_\alpha(r_0)|^2 \delta(\omega_f - \omega_\alpha),
\]  

(49)

where \( \omega_f = e_f - e_i \), \( e_i x_{fi} \) is the dipole matrix element and the index \( \alpha \) stands for the emitted photon parameters

\[
\alpha = \left\{ \begin{array}{ll}
 k_z, n, \ell, & \omega_\alpha = \omega_{n, \ell}(k_z), & \text{waveguiding photons,} \\
 k, \mu, & \omega_\alpha = |k|, & \text{traveling PW photons,} \\
 \mu, n, k_z, k_z, & \omega_\alpha = \sqrt{k_z^2 + k_z^2}, & \text{traveling CW photons.}
\end{array} \right.
\]

For the waveguiding radiation the above probabilities determine the energy emitted by a linear dipole oriented along a \( \hat{d}(x_{fi} = x_{fi} \hat{d}) \) direction, due to the emission of a waveguided photon specified by the index \( \{ n, \ell \} \):

\[
I_{n, \ell} = 2 \pi \frac{e^2}{\epsilon_0} x_{fi}^2 \int_{-\infty}^{\infty} d k_z \omega_{n, \ell}(k_z) \left| \hat{d} \cdot \hat{E}_{n, \ell}^w(r_0) \right|^2 \times \delta(\omega_f - \omega_{n, \ell}(k_z)).
\]  

(50)

As the waveguiding photons specified by a given value of \( k_z \) parameter can have a discrete set of permitted energies, compare Fig. 3, the above integral over \( k_z \) is either zero, when no photons of this type match the transition energy difference \( \omega_f \), or reduce to the single term

\[
I_{n, \ell} = 2 \pi \frac{e^2}{\epsilon_0} x_{fi}^2 \left| \frac{d}{dk_z} \omega_{n, \ell}(r_0) \right|^2,
\]  

(51)
where $\omega_{\text{fi}} = \omega_{n,i}(k_0)$. Notice that the above waveguiding photon contribution to the radiation intensity depends on the group velocity of the mode, $v_{gr} = \frac{\partial \omega_{n,i}}{\partial k_z}$.

For an open dielectric waveguide these group velocities are nonsingular [cf. Fig. 1(b)] and the above expression is always finite. This contrasts with the properties of a traditional waveguide made by a perfectly conducting pipe, investigated recently in [21]. The group velocities of the conducting waveguide modes near their thresholds vanish, causing divergencies of the emission rate. Such divergencies in the emission rate are usually attributed to the divergencies in "photon densities."

Performing calculations for a given waveguide and a given atomic transition frequency, it is convenient first to select the modes that can match the frequency condition. The number of such modes is always finite. Figure 2 shows this number as a function of the waveguide radius, and Figs. 1(a) and 1(b) show the corresponding propagation constants $\beta_{n,i}$ and $v_{gr}$ for all modes matching this condition. Summing over possible values of the angular partial numbers $n$ (positive and negative), sequential mode number $\iota$, and doubling the obtained value due to possible propagation of photons in the positive and negative directions of the $z$ axis, one gets the total contribution of the waveguiding photons to the radiation intensity and losses,

$$I^w(r_0) = 2 \sum_{n=-n_{\text{max}}}^{n_{\text{max}}} \sum_{\iota=0}^{\nu(n)} I_{\iota n}^w(r_0).$$

Now, the emission of traveling photons will be discussed. Using the plane-wave parametrization of the electromagnetic modes of the outgoing type, a radiation intensity flux of a polarization $\mu$ crossing a surface element $d\Omega = \cos \theta d\theta d\phi d\phi$ is given by

$$dI_{\mu}(k) = d\Omega \int dk k^2 \omega(k) P_{k,\mu},$$

and the radiation intensity distribution is

$$I_{\mu,\Omega} = \frac{dI_{\mu}}{d\Omega} = 2\pi \frac{e^2}{\epsilon_0} \omega_{\text{fi}}^{\mu} |x_{\text{fi}} \cdot \mathbf{E}_{k,\mu}(r_0)|^2.$$

Integrating $I_{\mu,\Omega}$ over the whole spherical angle and summing over both polarizations one gets the total radiation flux due to the traveling photons

$$I' = 2\pi \frac{e^2}{\epsilon_0} \omega_{\text{fi}}^{\mu} \sum_{\mu=1,2} \int d\Omega |\mathbf{d} \cdot \mathbf{E}_{k,\mu}(r_0)|^2.$$  

The direct evaluation of this integral presents some difficulties as the mode fields $\mathbf{E}_{k,\mu}$ are rapidly oscillating functions of $\phi$ and $\theta$ due to the interference between the plane-wave field $\mathbf{E}$ and the converging assembled field $\mathbf{E}^{(2)}$. In performing this computation it is more convenient to use traveling angular cylindrical photons. In this formulation the total emitted radiation is

$$I' = 2\pi \frac{e^2}{\epsilon_0} \omega_{\text{fi}}^{\mu} \sum_{\mu=1,2} \int d\Omega |\mathbf{d} \cdot \mathbf{E}_{k,\mu}(r_0)|^2.$$  

Although in the above expressions the summation over the partial waves, i.e., over $n$, extends from $-\infty$ to $+\infty$, the terms above certain $n$, (dependent on the dipole position $r_0$) give only marginal contributions as these waves are very small at the position of the dipole.

In this work two types of modes called in and out are recognized. These two types of modes are completely equivalent, nevertheless the outgoing modes and corresponding outgoing photons allow a simpler description of angular characteristics of emitted spontaneous radiation. Regarding the total intensity as given by Eqs. (54)–(57), it is not important which types of modes (i.e., in modes or out modes) will be employed. In order to obtain the angular pattern, however, it is the out modes that are to be inserted into Eqs. (53),(58). Indeed, if one were to use the in modes, one would have to take into account all possible interferences among these modes since the Poynting vector determining the flux towards the detector would contain the contributions from all such modes [10]. On the other hand, in the case of out modes used to calculate the angular pattern the Poynting vector contains the contribution from a single emergent wave only. This is why we have written the radiation intensity flux (52) in terms of the outgoing modes.

Convenience of use of the outgoing waves as the basis in scattering problems was noticed in connection with a distorted wave approximation, cf. [18]. Following this analogy, the dielectric cylinder and other insertions modifying the radiating atom’s environment results in such primary distortion of the free electromagnetic field, that the secondary pertur-
V. RESULTS AND DISCUSSION

The first example of the presented theory shows the rate of emission of the waveguided photons. The waveguide considered with radius $a = 2\lambda$ ($\lambda$ is determined by the frequency of the transition), and $\epsilon_d = 1.5$ has 44 different discrete propagation parameters (degenerated due to the sign of $k_z$ and $n$), representing 160 possible modes and photons excited in the waveguide. Their excitations depend on the source dipole position and its orientation. Figures 8(a) and 8(b) give the emission rate calculated according to Eq. (51) for two positions of the dipole inside the waveguide and their three orientations: axial along the $z$ axis (represented by the vertical lines), radial along the $r$ axis (represented by the horizontal lines) and azimuthal along the $\phi$ axis (represented by the skew lines). The length of these lines is proportional to the emission rate of the particular photon. The dipoles located near the waveguide center can excite only the low partial waves (those exactly at the center only $n = 0, \pm 1$ modes) and the higher $n$ modes are excited when the dipole is moved outside. Excitations of the waveguide modes are possible by the dipoles placed outside of the waveguide, however, these rates are rapidly decreasing when the dipole is moved away from the waveguide.

Let us now turn to the problem of emission into the traveling modes. The traveling modes are defined by the solutions of the scattering problem or an “inverse” problem, named here an assembling problem. The numerical methods that we use, with high accuracy Bessel functions, allow us to find exact numerical solutions of this old diffraction problem in the frame of the rigorous diffraction theory based on the Maxwell equations. In particular, these solutions show that a dielectric cylinder acts like a lens, concentrating and focusing incident plane waves, even for small radii of the cylinder when one may not expect the geometric optics concepts to be valid.

An example, corresponding to the asymptotic TM polarized plane wave, propagating along the $x$ axis perpendicularly to the cylinder axis, is shown in Fig. 4(a) as a three-dimensional (3D) surface plot and in Fig. 4(b) as a contour plot. The field is represented by the absolute value of the $E_z$ component. Focusing properties including details of the focus spot are clearly exposed. In particular, one can find that the field between the cylinder and the focus, with many local “hills and valleys,” is very different from the smooth field distribution behind the focus. This property is quite different from that valid in paraxial approximation, described in [23, 24]. The results presented in this paper are not restricted to the paraxial approximation and the structure found in the focus may be a consequence of a strong cylindrical aberration and rather poor focusing property of a cylindrical lens.

Despite this defect of the dielectric cylinder as a high quality lens, we can still use this model to reveal how the focusing properties influence the spontaneous emission.

The modes shown in these pictures can be interpreted either as the incoming modes or the outgoing depending on whether one has in mind the scattering problem or the assembling problem. In the in-mode case they represent the field distribution induced by the plane wave, incident along the $x$ axis from the left, from $-\infty$ superposed with the scattered wave, diverging outward from the cylinder to $r \to \infty$. In the out-mode case one must interpret this plane wave as emergent along the $x$ axis to the left, to $-\infty$ while the associated assembled wave is converging from $r = \infty$ toward the center at $r = 0$.

Figure 9 shows an angular pattern of the emission rate in the $x$-$y$ plane around the cylinder, calculated according to Eq. (53), for four positions of the dipole. The parameters of the system are the same as in Fig. 4 and the dipole is oriented along the $z$ axis. For a fixed position of the dipole the mode relevant for the emission in a given direction $\phi$ in the $x$-$y$ plane can be obtained by an appropriate rotation of the single mode presented in Fig. 6. Alternatively, one may fix the mode and rotate the dipole around the $z$ axis.

Thus for the dipole placed at the focal distance ($r_0 = 5$ in the presented example) the azimuthal distribution of radiation intensity concentrates in a narrow peak. This peak is surrounded by a sector of a very low radiation intensity, beyond which an average intensity remains the same as in free-space situation, showing, however, rapid angular variations resembling random fluctuations. When moving the dipole closer toward the cylinder the peak splits into several, closely spaced smaller peaks, while when moving it away the focused radiation peak is decreasing, spreading, and eventually vanishing.
More examples of the angular radiation patterns, dependent on both spherical angles $\phi$ and $\theta$ for various orientations of the dipole and radiation polarizations, are shown in Fig. 10. Figure 10(a) shows what the concentrated and focused radiation in the horizontal plane looks like and spreads for growing $\theta$. Figure 10(d) shows that the dipole oriented along the $z$ axis can emit the TE polarized radiation, with the electric field laying in the $x$-$y$ plane. This can only happen for $\theta \neq 0$ illustrating a hybrid nature of the modes in the vicinity of the dielectric cylinder.

The examples presented hitherto show a rather regularly varying radiation pattern, despite rapid oscillations in $\phi$ shown in Figs. 9 and 10. The density of polar intensity $I_{\phi}(\theta)$ given by Eq. (58) and shown in Figs. 11(a) and 11(b) exhibits, in addition to this regular pattern, some number of very narrow and high in magnitude extra spikes appearing rather irregularly. These narrow spikes, omitted simply in Figs. 9,10 because of rare samplings of the emission data, are due to the presence of sharp resonances in the set of the traveling modes. These resonances, known for a long time as whispering gallery modes, and widely investigated and described mostly for dielectric spheres, cf. [25,20] and the references quoted there, present a real challenge in the radiation emission theory.

The main problem is that these resonant photons are strongly coupled to the atoms and, in addition, the coupling terms are extremely rapidly varying violating the conditions required by the perturbative approach and the validity of the Fermi golden rule.

There is some hope that the perturbative approach can be rescued due to the narrowness of these resonances and, overall, their small contribution to global features of the radiative decay. They may play an important role in the detail properties of the emission line shape. This opinion finds a confirmation in the nonperturbative solutions of some simplified models which include the strong coupling conditions, [26,27]. In this paper we do not attempt to describe these fine effects and therefore the discussion will still be restricted to the golden rule, remembering, however, about the warnings pointed out above.

The determining parameters of the whispering gallery modes, i.e., their characteristic partial wave number $n$ and
their widths are shown in Figs. 12(a),12(b). Figure 12(a) shows the resonant scattering angles \( \theta_{res} = \arcsin(\text{Re } \beta_{res}) \), and the partial wave indices \( n \) indicating the particular resonance, while Fig. 12(b) shows the width of resonance given by \( \text{Im } \beta_{res} \). These data allow us to determine the partial wave index \( n \) and the sequential index \( \ell \) of all resonant peaks shown in Figs. 11(a) and 11(b).

In Fig. 13 we show the decay constant \( \Gamma \), i.e., the total radiation intensity \( I' \) normalized to that of free space, for three diameters of the waveguide and three dipoles orientations as a function of the dipole displacement from the waveguide center. These plots compare contributions of the waveguiding and traveling photons, including their polarizations. It appears that the most efficient emission corresponds to the radiating dipole with radial orientation placed just outside the waveguide. This emission rate is discontinuous across the waveguide boundary due to the discontinuity of the radial component of the electric field \( E_r \).

The emission of the dipole placed inside the waveguide and surrounded by the atoms making the dielectric can be modified due to corrections coming from local field effects. These corrections could be accounted for in a bulk isotropic dielectric by a Clausius-Mossotti factor \( 3\varepsilon_d/(2\varepsilon_d+1) \) multiplying the strength of the mean electric field responsible for the dipole-radiation coupling [11,3].

VI. FINAL REMARKS

In this paper we have investigated the radiation of an atom near a dielectric cylinder. We believe that the following points of our discussion are quite remarkable. Firstly, the canonical quantization of the electromagnetic field in the presence of the dielectric lossless cylinder have been performed using the full mode decomposition of the field. Various parametrizations of the mode functions have been introduced, among them, the parametrization by outgoing cylindrical waves have been found to be most suitable for describing spontaneous emission from the atom. Second, the connection between the waveguiding modes and the whispering gallery modes for the case of a dielectric cylinder has been expounded and made transparent. Third, the spontaneous decay rates have been obtained for an arbitrary location of the atom with respect to the dielectric cylinder and arbitrary polarization of the atomic transition moments. The contributions of the trapped and of the scattered modes to the total rate have been specified. Limitations of the standard approach, which is based on the Fermi golden rule due to the existence of rapidly changing parts of the electromagnetic field strongly coupled to atoms, have been pointed out. Fourth, the detailed description of the far-field emitted by the atom near the dielectric rod has been given. The radiation pattern of the field is the same in the classical and quantum cases and hence our discussion of radiation is equally valid.
for the classical field in far zone produced by a point source near the cylinder. Fifth, the problem of the field behavior near the focus when a plane wave is scattered by the cylinder has been studied. Our exact numerical approach has allowed us to indicate severe limitations of the paraxial approximations, presumably due to the strong aberration of the cylindrical lens. It is to be mentioned here that the spontaneous emission rate itself is a measure of behavior of electromagnetic modes in the near wave zone, i.e., in the proximity of the cylinder.

It is interesting that the part of the field that includes the waveguiding modes does not show any singularities. In consequence, the properties of the dielectric waveguides, and the conventional waveguides, perfectly conducting tubes (in which singularities occur at the waveguide thresholds), are very different. The differences between two types of waveguides—conducting and dielectric—become very transparent when one compares the group velocity of the guided modes, which can vanish in the case of metallic guides but never vanishes in the case of dielectric ones. The strength of the coupling of atoms with the field is often expressed in terms of the density of modes. Here, there was no need to introduce this concept.

The field modes themselves in both systems are regular, i.e., they can be properly normalized and are nonsingular functions of their parameters, though these functions could be rapidly varying. Singularities rather occur in the traditional perturbative approaches, requiring a slow evolution of the system under consideration. In some cases, e.g., for atoms in metallic waveguides interacting with the field modes near their thresholds, this evolution is not so slow, and other, nonperturbative methods, are necessary, see, e.g., [26–28].

In dielectric waveguides, although there are threshold frequencies for different modes, they do not contribute to any violation of the perturbative approach. However, in these waveguides the atoms can be strongly coupled to the whispering gallery resonances occurring in the traveling part of the field modes. These resonances, however, do not distinguish any characteristic frequencies, unlike the perfectly conducting waveguides at thresholds and unlike the resonant cavities.

The whispering gallery resonances can occur in frequency bands, specific for each of the partial modes. They are characterized by a particular relation between the longitudinal component of the wave vector \(k_z\) and the mode frequency \(\omega\). The atom, interacting with a continuum of the field modes, can always find resonant conditions, however, in extremely narrow ranges of parameters. The estimation of their effects is much more difficult than in the perfectly conducting waveguides at thresholds. It is the most challenging question in these problems.

The system analyzed in this paper, i.e., an atom and electromagnetic fields in the presence of a dielectric waveguide is very important both in the pure research within the framework of QED and in the technical applications. This system includes different types of photons—waveguiding and traveling ones. The traveling modes should be classified in two equivalent bases, one of them connected with the parametrization based on the plane waves, the other one based on cylindrical waves. The former basis is useful and important to obtain the angular distribution of radiation, the latter one to analyze the global properties of the atom-radiation interactions like the decay rate.

We believe that the results described here can and should be compared with experiment. For instance, the experimental methods of implanting atoms on a dielectric surface as developed for planar waveguides and presented by Snoeks and coworkers [7] can be applied to the cylindrical waveguide as well. This would enable verification of our predictions regarding the excitation of waveguide modes, the angular patterns, and emission rates obtained in this paper. An experiment of the type proposed here would provide necessary data to develop a theoretical description of the interaction of atoms with shape resonances, which is still waiting for more complete investigations.

**ACKNOWLEDGMENTS**

We would like to thank Professor Janusz Dąbrowski for valuable comments on the normalization problem. This work has been supported by the Polish State Committee for Scientific Research Grant No. 2 P03B 030 13.

**APPENDIX: ORTHOGONALITY AND NORMALIZATION OF TRAVELING CYLINDRICAL MODES**

In the Appendix we provide the proof of the orthogonality relations between various modes. Our approach is the generalization of the method of proving the orthogonality properties of wave functions in quantum mechanics, outlined in [29], to the case of electromagnetic waves in Maxwell’s theory. It is to be noted that the proof reduces itself to the proof of orthogonality of modes corresponding to different frequencies. In many cases the latter relation is considered as self-evident. However, we shall provide explicit arguments in favor of this assertion.

Let us consider two collections of the solutions of the stationary Maxwell equations: \(\{\mathbf{E}, \mathbf{B}\}\) corresponding to the frequency \(k_0\), and \(\{\mathbf{E}', \mathbf{B}'\}\) corresponding to the frequency \(k'_0\). It follows from the Maxwell equations that

\[
i(k_0 - k'_0)(\mathbf{eE}'\cdot \mathbf{E} + \mathbf{B}'\cdot \mathbf{B}) = - \nabla \cdot (\mathbf{E} \times \mathbf{B}' + \mathbf{E}' \times \mathbf{B}).
\]

(A1)

Let \(S(R)\) denote a very large cylindrical surface with the radius \(R\) coaxial with the dielectric rod, and let \(V(R)\) be the volume contained within the surface. From the Gauss theorem one finds

\[
G(R) = \int_{V(R)} d^3r (\mathbf{eE}'\cdot \mathbf{E} + \mathbf{B}'\cdot \mathbf{B}) = - \frac{1}{i(k_0 - k'_0)} \int_{S(R)} d\sigma \mathbf{n}_s \cdot (\mathbf{E} \times \mathbf{B}' + \mathbf{E}' \times \mathbf{B}).
\]

(A2)

where \(\mathbf{n}_s = \hat{\mathbf{r}}\) and \(d\sigma = Rd\phi dz\).
The scalar product of two such modes does not vanish identically. The function $F_{k_z,k_\perp,k_z',k_{\perp}'}(R)$ is an oscillating function of $R$ (for given $k_0$ and $k_0'$) and of the difference $(k_0-k_0')$ for given $R$. In these latter oscillations, however, there is no characteristic central peak that is responsible for the Dirac-$\delta$ like behavior of $F$ for two identical polarizations, see Fig. 14. Therefore, an integral of any well-behaved test function between the modes corresponding to different polarization.

Due to the unitarity condition [cf. Eq. (27)] applied to TM modes

\[
|D_n^{12}|^2 + |C_n^{12}|^2 = \frac{E_0^2}{4},
\]

using asymptotic expansion of the Hankel functions, and taking

\[
\frac{1}{k_0-k_0'} \frac{k_0}{k_\perp-k_{\perp}'}
\]

one gets in the limit $R \to \infty (E_z^0=E_z k_\perp/k_0)$.

\[
\lim_{R \to \infty} F_{k_z,k_\perp,k_z',k_{\perp}'}^{11}(R) = \lim_{R \to \infty} \frac{2k_0^2 E_0^2 \sin(k_\perp' - k_\perp)}{k_\perp - k_{\perp}'} \frac{1}{k_\perp}.
\]

Taking $E_0^2=1/\sqrt{2}$ we obtain the standard normalization condition used in this paper. A very similar procedure leads to the proof of orthogonality among TE modes.

Let us now briefly address the problem of orthogonality between the modes corresponding to different polarization. The scalar product of two such modes does not vanish. The function $F_{k_z,k_\perp,k_z',k_{\perp}'}^{12}(R)$ is an oscillating function of $R$ (for given $k_0$ and $k_0'$) and of the difference $(k_0-k_0')$ for given $R$. In these latter oscillations, however, there is no characteristic central peak that is responsible for the Dirac-$\delta$ like behavior of $F$ for two identical polarizations, see Fig. 14. Therefore, an integral of any well-behaved test function of $k_0-k_0'$ multiplied by $F^{12}$ approaches zero when $R \to \infty$.

We conclude that $F^{12}$ is equal to zero in the distributional sense.

Let us notice that Eq. (27) can be applied for the waveguided modes as well. In this case the orthogonality
results immediately since the waveguided modes vanish for $R \to \infty$ sufficiently fast being square integrable in the radial variable.

Finally, a scalar product of a waveguided mode and a traveling modes must include $\delta(k_z - k'_z)$. Hence it must be zero because if they have the same frequency $k_0$ they differ in $k_z$, which is greater than $k_0$ for the waveguided mode and smaller than $k_0$ for the traveling mode.


