Soluble model of many interacting quantum particles in a trap

Magdalena A. Zaluska-Kotur, Mariusz Gazda, Arkadiusz Orłowski, and Jan Mostowski
Instytut Fizyki PAN & College of Science, Aleja Lotników 32/46, 02-668 Warszawa, Poland

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Exact solutions to many-body interacting systems of both bosonic and fermionic particles confined to harmonic potential in an arbitrary number of dimensions are given. Energy levels and their degeneracies for trapped identical particles interacting via harmonic forces are calculated. This specific form of the interaction allows for analytical solutions. The mutual interaction, attractive or repulsive, modifies significantly the properties of the considered system. For a large number of particles the interaction essentially results in a frequency shift. Statistical properties (e.g., microcanonical and grand canonical partition functions) as well as some illustrative, physically relevant examples are discussed. Our results give an unusual opportunity for further studies of interacting systems in the framework of the exactly soluble model.

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I. INTRODUCTION

Quantum many-body theory is a mature [1] yet dynamically developing subject of intensive studies [2]. Recent achievements in the physics of trapped neutral atoms have triggered more interest toward areas left unexplored till now. Techniques have been developed to cool trapped atoms down to temperatures where pure quantum effects connected not only with electrons but with whole atoms are clearly visible. Progress in atomic traps now allows us to confine big clouds of atoms (of the order of $10^7$ atoms) cooled down to nanokelvin temperatures. In most experiments alkali atoms are used as trapped objects. Different species can exhibit either Bose-Einstein statistics (such as sodium $^{23}$Na, rubidium $^{85}$Rb, and lithium $^7$Li atoms) or Fermi-Dirac statistics (such as $^6$Li or $^{40}$K). On the theoretical side the understanding of Bose gases is rather good. The most dramatic effect, namely, Bose-Einstein condensation is well understood [3]. Proper and full incorporation of effects depending on quantum statistics is needed to describe the effect as well as other properties of the Bose gas. Physics of ultracold Fermi gases is not so well developed as that of Bose gases. The reason is that evaporation cooling, so successful in the bosonic case, is not applicable for Fermi atoms; it is even harder to achieve nanokelvin temperatures. Fortunately the future of experiments with cold Fermi gases looks more promising today due to the development of sympathetic cooling. This method was first devised to cool down different species of bosons (binary condensates) but can also be effectively used for fermions [4].

A cloud of Fermi or Bose atoms constitutes a weakly interacting quantum gas. Because of the external trapping potential these gases are intrinsically inhomogeneous. Interaction between atoms makes the system difficult to study theoretically. Several approximation schemes have been developed to provide description of such gases. In case of bosons the most widely used are based on the Gross-Pitaevski equation and on the Bogoliubov method of separation of the condense part of the gas from the excitations. In this paper a different approach is used. We introduce approximate interaction potential between particles in such a way that the many-body problem becomes exactly soluble. The model used here has its roots in the early days of nuclear physics and it has been rediscovered many times ever since [5]. Our interest originates from the recent experiments with cold trapped bosonic and fermionic systems. Although there is some overlap between our results and those recently published [6], approach presented here is essentially different. The mentioned papers used path integration techniques and manage to evaluate some basic physical properties for relatively large number of particles. Here a second-quantization unified approach, encompassing both fermions and bosons within a single formalism, is advocated. Essential features of the diagonalization procedure do not depend upon the particle statistics. This provides more physical insight and clarifies many of the features found in the literature. Moreover in our paper we provide explicit formulas for the wave functions of the interacting many particle system. In addition we were able to find microcanonical partition function for the interacting system and express it in terms of the corresponding noninteracting system. Very effective analytic and numerical methods have been recently developed to deal with the latter problem in the context of the Bose-Einstein condensation [7–9]. This allows to find in a straightforward way the degeneracies of the energy eigenstates.

The harmonic potential interaction used in this model has some peculiar properties. The main one is that the force between particles forming the gas is increasing with distance between the particles. Such interaction should not be used in the case of homogeneous systems because the total interaction energy would be infinite. In the case of nuclei as well as in our case of trapped particles the system is small, separation between particles is limited by external forces and, therefore, the long range of interaction is of no serious danger. Another thing is how accurately do long range interactions describe real systems. Standard approaches, such as the Gross-Pitaevski equation are based on a different approximation, namely, that the interactions are of short range (in fact of zero range). Therefore the long range model of interaction used in our approach can be considered only as a crude approximation of real systems. Still it has several advantages. The first one is that the model is exactly and analytically soluble. Such models are of importance for their own sake. As such it can be used as a theoretical vehicle for testing...
various approximations that are indispensable in the analysis of more realistic situations: more physical insight can be gained into the physics of many body phenomena [10]. Our formalism is capable to deal with both attractive and repulsive interaction on equal footing.

The paper is organized as follows. In Sec. II we formulate the model. In Sec. III some illustrative examples are given. Small number of interacting particles makes it possible to write explicit solutions in terms of many particle wave functions. Systematic solution for arbitrary number of particles is presented in Sec. IV. The form of grand canonical partition function for interacting system derived in Sec. V shows that most of thermodynamic properties are similar to those of the noninteracting system provided the energy is properly rescaled. Two-point correlation functions are compared for systems of interacting and noninteracting particles. We finish in Sec. VI with concluding remarks.

II. THE HARMONIC INTERACTION MODEL

Let us consider a system of \( N \) identical quantum particles (fermions or bosons), in an external harmonic potential. The particles interact with each other by potential forces. The potential is taken in the form of a quadratic function of the coordinates. As explained in the Introduction such a choice makes the model soluble. On the other hand harmonic potential is nonphysical as it leads to forces that grow with increasing distance between the particles. Therefore the model cannot be used for spatially homogeneous systems since infinite energies would apparently occur. For finite systems bounded by external potential the model does not lead to any divergences. In fact it has successfully been used in atomic physics [11].

Let \( \mathbf{x}_1, \ldots, \mathbf{x}_N \) denote the positions of particles and let \( \mathbf{p}_1, \ldots, \mathbf{p}_N \) stand for canonical momenta of the particles. Particles are confined by external harmonic potential and interact with themselves via harmonic interactions with the coupling characterized by frequency \( \Omega \). We use units in which mass of the particles as well as the frequency of the external harmonic potential are equal to 1. The total Hamiltonian reads

\[
H = \frac{1}{2} \sum_{i=1}^N (\mathbf{p}_i^2 + \mathbf{x}_i^2) + \frac{1}{2} \Omega^2 \sum_{i=1}^N \sum_{j<i} (\mathbf{x}_i - \mathbf{x}_j)^2. \tag{1}
\]

It is useful to transform this Hamiltonian to the form

\[
H = \frac{1}{2} (\mathbf{q}_1^2 + y_1^2) + \sum_{i=2}^N (\mathbf{q}_i^2 + \omega^2 y_i^2) = H^C + H^{rel}, \tag{2}
\]

where \( \omega \) depends on the interaction strength and the number of interacting particles

\[
\omega^2 = 1 + N \Omega^2. \tag{3}
\]

Variables \( y_1 = 1/\sqrt{N} \sum_{i=1}^N x_i \), and \( q_i = 1/\sqrt{N} \sum_{i=1}^N p_i \), correspond to the center of mass; \( y_i \) for \( i = 2, \ldots, N \) constitute a set of orthogonal variables that span the subspace orthogonal to the vector \( y_1 \), and \( q_i \) are canonical momenta conjugate to \( y_i \). The \( \pm \) signs correspond to attractive and repulsive interactions, respectively.

Apparently the Hamiltonian in formula (2) is a sum of two terms. The first, that describes the center of mass motion with characteristic frequency equal to one, and the second corresponding to relative degrees of freedom with energy unit \( \omega \). The center of mass part depends on one variable only and it has a simple energy spectrum of harmonic oscillator: equally spaced levels with eigenfunctions depending on the variable \( y_1 \). On the other hand the eigenvalue problem for \( H^{rel} \) is much more complicated. Eigenenergies look as if they were separated by constant frequency \( \omega \) and eigenfunctions as if they were a product of single particle oscillatory functions. This is, however, not the case. Symmetry of the wave function eliminates some of the states. Therefore, though the interacting problem can be reduced to the noninteracting case, the spectrum is somewhat different from that expected for a simple harmonic oscillator. This will also significantly influence thermodynamic properties of the system under consideration.

We are ready now to diagonalize the full Hamiltonian and find the degeneracies of the states. However, before we proceed with general discussion let us first consider one-dimensional systems with small number of particles. This will illustrate mechanisms by which particle statistics influences the spectrum.

III. SMALL NUMBER OF PARTICLES IN ONE DIMENSION—AN ILLUSTRATIVE EXAMPLE

Consider two particles, fermions or bosons, in one-dimensional harmonic external potential, interacting between themselves via harmonic forces. We introduce center of mass \( y_1 = (x_1 + x_2)/\sqrt{2} \) and relative \( y_2 = (x_2 - x_1)/\sqrt{2} \) coordinates as well as two length scales: \( l_0 = 1 \), corresponding to the width of the ground state without interactions and \( l_1 = \omega^{-1/2} \) corresponding to the width of the ground state of harmonic oscillator with frequency \( \omega \). Eigenenergies of the interacting system are

\[
E_{n,m} = (n + 1/2) + \omega (m_0 + m + 1/2), \tag{4}
\]

where \( m_0 = 0 \) for bosons and \( m_0 = 1 \) for fermions and \( n, m \) are nonnegative integers. The wave function can be easily found:

\[
\psi_{n,m}(x_1, x_2) = N H_n(y_1) H_m(y_2/l_1) \exp(-y_1^2/2 - \omega y_2^2/2), \tag{5}
\]

where \( N \) is normalization factor and \( H_n \) denotes Hermite polynomials. The wave function of bosons should be symmetric, therefore \( m \) should be even; for fermions the wave function is antisymmetric and \( m \) must be odd. In both cases the spectrum of relative excitation contains “half” of states, i.e., the separation between energies is \( 2 \omega \) rather than \( \omega \).

The case of three particles is even more interesting. The ground state wave function of bosons (up to normalization factor) is given by
while for fermions the ground state (again up to normalization) takes the form
\[
\psi_0^F(x_1,x_2,x_3) = (x_1-x_2)(x_2-x_3)(x_3-x_1) \psi_0^F(x_1,x_2,x_3).
\] (7)

Excitation of the system leads to the following two lowest fermionic states
\[
\psi_{2,0}^F(x_1,x_2,x_3) = \psi_0^F(x_1,x_2,x_3)(x_1^2 + x_2^2 + x_3^2 + 3x_1x_2 + 3x_2x_3 + 3x_3x_1),
\] (8)
\[
\psi_{0,2}^F(x_1,x_2,x_3) = \psi_0^F(x_1,x_2,x_3)(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1 - 6).
\] (9)

The first state corresponds to excitation of relative coordinates by the two quanta and no center of mass excitation, the second one to double center of mass excitation. In case of three particles the spectrum of internal excitations is as follows. The state of energy \(\omega\) above the ground state is missing, the next states with energies \(2\omega\) and \(3\omega\) are present and they are not degenerate. However, all other states with higher energies are degenerate.

In general, eigenstates of the Hamiltonian (1) for an arbitrary number of identical particles can be written in the form
\[
\psi(x_1, \ldots, x_N) = \sum C_{n_1, \ldots, n_N} \psi_{n_1}(y_1) \prod_{i=2}^N \psi_{n_i}(y_i),
\] (10)

where \(\psi_{n_i}\) are single particle oscillatory wave functions. The coefficients \(C_{n_1, \ldots, n_i}\) must be chosen in such a way that the total wave function has required symmetry (boson or fermion type) with respect to exchange of variables \(x_1, x_2, \ldots, x_N\). Proper symmetrization of the wave function is an essential issue in the quantum description of many-body problems.

IV. EXACT SOLUTION TO EIGENPROBLEM

A. Second quantization picture

Examples considered in the previous section do not allow for straightforward generalization to larger number of particles. They show that symmetry constraints lead to serious difficulties. The most convenient way to incorporate the symmetry of a wave function with respect to particles exchange is to work in the second quantization picture. Construction presented below will demonstrate directly that our method is completely general and applicable to any spatial dimension \(d\). Let us define field operator (as well as its Hermitian conjugate)
\[
\Psi(x) = \sum_k \phi_k(x) a_k,
\] (11)

where \(a_k^\dagger\) creates \((a_k\) annihilates\) particle at the eigenstate \(\phi_k(x)\) of the harmonic oscillator of frequency \(\omega\). Index \(k\) means a set of \(d\) independent oscillators \(\{k_1, \ldots, k_d\}\). Creation and annihilation operators obey the usual commutation rules for bosons and anticommutation rules for fermions. Operators \(\hat{N}_k|n\rangle = a_k^\dagger a_k |n\rangle\) and \(\hat{N} = \sum_k a_k^\dagger a_k\) correspond to the occupation of a given single particle oscillatory level \(k\) and to the total number of particles, respectively. The eigenvectors \(|n\rangle = |n_0, n_1, \ldots\rangle\) of all occupation operators
\[
\hat{N}_k |n\rangle = n_k |n\rangle,
\] (12)
span the Hilbert space for second quantization operators. From now on we restrict our analysis to the subspace of definite number of particles, i.e., to these states \(|\bar{n}\rangle\) of the Hilbert space which fulfill the additional condition
\[
\sum_k n_k|\bar{n}\rangle = N|\bar{n}\rangle.
\] (13)

Occupation number can be any nonnegative integer in the case of bosons \((n_k = 0,1,2, \ldots)\) while for fermions (due to the Pauli exclusion principle) it can have only two values \((n_k = 0,1)\) depending if a state is occupied or not.

In subspace of definite number of particles we can separate the Hamiltonian (1) into two parts: the center of mass and the relative degrees Hamiltonian [see Eq. (2)]. Note that both these Hamiltonians involve single particle as well as two particle interaction terms. For example, the center of mass part written in the configuration space
\[
\hat{H}_C = \frac{1}{2N} \sum_{i,j} (p_i^2 + x_i^2) + \frac{1}{2N} \sum_{i,j} (p_i p_j + x_i x_j)
\] (14)
can be transformed into the second quantization form according to the standard prescription
\[
\hat{\Psi}^C = \frac{1}{2N} \int dx \Psi^\dagger(x)(-\nabla_x^2 + x^2) \Psi(x) + \int dx dx' \Psi^\dagger(x) \Psi^\dagger(x') \times \left((\nabla_x \nabla_{x'} + x x') \Psi(x) \Psi(x')\right).
\] (15)

In a similar fashion we express the total Hamiltonian in terms of bosonic (or fermionic) creation and annihilation operators which obey the usual commutation rules for bosons and anticommutation rules for fermions. That ensures required symmetry of the eigenstates. The second quantization form of the total Hamiltonian is
\[
\hat{H} = \hat{H}_C + \hat{H}_R,
\] (16)

where \(\hat{H}_R\) is relative degrees excitation operator,
\[
\hat{H}_R = \frac{d}{2} \left( \sum_{l=1}^d \sum_{i,j} C_{w_i}^\dagger c_{w_j}^\dagger + \frac{d}{2} \right),
\] (17)
and
\[
\hat{H}^C = \sum_l C_l^\dagger C_l + \frac{d}{2},
\]  
(18)
where \( l = 1, \ldots, d \). In the following we will subsequently define all operators appearing in the above equations while simultaneously commenting on their essential properties.

The Hamiltonian describing the relative degrees of freedom \( \hat{H}_0^\text{rel} \) is a sum of two terms. The first one \( \hat{H}_0 \) is simply the Hamiltonian of noninteracting particles in the harmonic potential of frequency \( \omega \).

\[
\hat{H}_0 = \omega \sum_k \left( \sum_l k_l + \frac{d}{2} \right) a_k^\dagger a_k,
\]  
(19)
while the second term, \( C_l^\dagger C_{\omega l} \), measures the number of the center of mass excitation quanta (in units of \( \omega \)) of this noninteracting system. These excitations have bosonic character (independently on the particle statistics) as they are annihilated by the operators

\[
C_{\omega l} = \frac{1}{\sqrt{N}} \sum_k \sqrt{k} a_k^\dagger, \ldots, k_{l-1}, \ldots, k_d a_{k_1}, \ldots, k_l.
\]  
(20)

independently for each spatial direction \( l \). They obey bosonic commutation relation with their conjugated partners

\[
[C_{\omega l}, C_{\omega l}^\dagger] = 1.
\]  
(21)

It is not surprising that noninteracting Hamiltonian commutes with its own center of mass part, \( [\hat{H}_0, \Sigma C_{\omega l}^\dagger C_{\omega l}] = 0 \), therefore both these operators have a common set of eigenvectors. Commutation relations

\[
[\hat{H}_0, C_{\omega l}] = -\omega C_{\omega l},
\]  
(22)
show that \( C_{\omega l} \) are the lowering and \( C_{\omega l}^\dagger \) are the raising operator for the Hamiltonian \( \hat{H}_0 \).

The term (18) is the genuine center-of-mass Hamiltonian whose unit of excitation energy is equal to 1 [compare Eq. (1)]. The excitations of this collective degrees of freedom are annihilated by the operators \( C_l \). This operator is related to the previously defined bosonic operators \( C_{\omega l} \) and \( C_{\omega l}^\dagger \) by the Bogolubov transformation

\[
C_l = \frac{1}{\sqrt{\omega}} + \frac{\sqrt{\omega}}{\sqrt{\omega}} C_{\omega l}^\dagger + \frac{1}{\sqrt{\omega}} - \frac{\sqrt{\omega}}{\sqrt{\omega}} C_{\omega l},
\]  
(23)
which indicates that oscillatory interaction between particles leads to a squeezing in the subspace of relative degrees of freedom. Because of the commutation relation

\[
[\hat{H}_0, \hat{H}_0^C] = 0,
\]  
(24)
both Hamiltonians have common set of eigenvectors.

B. Diagonalization of relative Hamiltonian

We start with the diagonalization of the relative Hamiltonian \( \hat{H}_0^\text{rel} \) first. Note that Hamiltonian \( \hat{H}_0 \) is diagonal in the natural basis (12) of the Hilbert space. All vectors fulfilling the condition \( \Sigma k_l n_l = m \) correspond to the same energy of \( \hat{H}_0 \). The ground state is defined by the lowest possible value of energy quanta \( m = m_0 \). In the ground state all noninteracting bosons occupy one state of the lowest possible energy, therefore \( m_0 = 0 \). Fermions are densely packed in the lowest \( N \) states, each fermion in one state. In the case of one-dimensional (1D) harmonic potential the number of energy quanta shared by \( N \) fermions in the ground state equals to \( m_0 = N(N-1)/2 \). In higher dimensions the fermionic ground state is, in general, degenerated.

All vectors, \( |m_e, \alpha\rangle \), of a fixed number of the excitation quanta, \( m_e = m - m_0 \), are the eigenstates of \( \hat{H}_0 \):

\[
\hat{H}_0 |m_e, \alpha\rangle = \omega \left( \frac{dN}{2} + m_0 + m_e \right) |m_e, \alpha\rangle,
\]  
(25)

\( m_e = 0, 1, 2, \ldots \), and \( \alpha \) enumerates different eigenstates. Let \( \Gamma_0^\mu(m_e) \) denotes degeneracy of states characterized by total number of excitation quanta \( m_e \). This excitation is shared by \( N \) identical noninteracting particles trapped in the external \( d \)-dimensional harmonic potential. Degeneracy depends on the energy level and also, more importantly, on the particle statistics; index \( \mu \) indicates particles statistics, \( \mu = B \) for bosons and \( \mu = F \) for fermions. The degeneracy is equal to the corresponding microcanonical partition function. Calculation of \( \Gamma_0^\mu(m_e) \) requires solving a very difficult combinatorial problem. In the case of bosonic atoms in the 1D oscillatory trap finding microcanonical partition function is identical with the famous number-theory problem of computing the number of unrestricted partitions of positive integer, solved by mathematicians in 1918 [12]. In 3D case the problem can be very effectively solved numerically for moderate system (about 1000 atoms) of bosons [7,8]. There is one to one correspondence between bosonic and fermionic microstates in 1D trap [13] while in higher dimensions the numerical methods mentioned above can be applied to fermions as well (with minor modifications only).

The crucial part of the whole diagonalization procedure is to diagonalize the relative degrees Hamiltonian in the basis formed by eigenstates of \( \hat{H}_0 \). To this end we find all these eigenstates of \( \hat{H}_0 \):

\[
|0_\omega, m_e, \alpha\rangle = \sum_{\alpha} b_{\sigma, \alpha} |m_e, \alpha\rangle,
\]  
(26)
where coefficients \( b_{\sigma, \alpha} \) are to be determined. These states should have no center-of-mass excitations, i.e., should be annihilated by all operators \( C_{\omega l}^\dagger \):

\[
C_{\omega l}^\dagger |0_\omega, m_e, \alpha\rangle = 0.
\]  
(27)

The symbol \( 0_\omega \) indicates the ground state of the operator \( \Sigma C_{\omega l}^\dagger C_{\omega l} \) (with eingenvalue equal to zero) and \( \sigma \) enumerates independent solutions of the Eq. (27). The number of
these solutions will be discussed in the following subsection. Because the vector \( C\omega \mid 0_{\omega} , m, \sigma \rangle \) is an eigenstate of the \( \hat{H}_0 \) corresponding to \( m - 1 \) excitation quanta [see Eq. (22)], the equations (27) may be transformed into set of linear equations for unknown coefficients \( b_{\sigma, \alpha} \):

\[
\sum_{\alpha} (\alpha', m_e - 1) \mid C_{\omega, l} \rangle m_{e, \alpha} b_{\sigma, \alpha} = 0, \tag{28}\]

where matrix elements of the operator \( C_{\omega, l} \) can be obtained directly from the definition, Eq. (20), and \( \alpha' = 1, \ldots, \Gamma^{\mu}_d (m_e - 1) \). This set of equations can be easily solved for \( d = 1 \), as in this specific case all equations are independent. For larger \( d \) operators for different space directions can lead to the same relations, hence some of equations (28) become linearly dependent. In order to find the proper eigenfunction in higher dimensions it is easier sometimes to diagonalize the operator \( \sum_{l} C_{\omega, l}C_{\omega, l} \) instead of solving Eq. (28). The energy spectrum that results from Eq. (28) will be discussed later.

The operator \( C_{\omega, l}^\dagger \) creates excitations of the center of mass (labeled \( n_{\omega, l} \)), therefore eigenvectors of both noninteracting and center-of-mass Hamiltonian are given by

\[
\mid n_{\omega}, m_e, \sigma \rangle = \prod_{l} \frac{(C_{\omega, l}^\dagger)^{n_{\omega, l}}}{\sqrt{n_{\omega, l}!}} \mid 0_{\omega, m_e - n_{\omega, \sigma}} \rangle, \tag{29}\]

with \( n_{\omega} = \sum_{l} n_{\omega, l} \). This way we have brought to the diagonal form the Hamiltonian of relative degrees of freedom:

\[
\hat{H}^{rel} \mid n_{\omega}, m_e, \sigma \rangle = \omega \left( d \cdot \frac{N - 1}{2} + m_0 + (m_e - n_{\omega}) \right) \mid n_{\omega}, m_e, \sigma \rangle. \tag{30}\]

C. Eigenstates of the total Hamiltonian and degeneracies

To fully determine the spectrum of the \( \hat{H}^{rel} \) we should also determine the degeneracies of all energy levels. Those, however, depend on the spatial dimension. Let us denote by \( g_d (n_{\omega}) \) the single particle degeneracy factor of \( n_{\omega} \)-th level of the \( d \)-dimensional isotropic harmonic oscillator. We remind the reader that \( g_1 (n_{\omega}) = 1 \), \( g_2 (n_{\omega}) = n_{\omega} + 1 \), and \( g_3 (n_{\omega}) = (n_{\omega} + 1)(n_{\omega} + 2)/2 \).

The states \( \mid n_{\omega}, m_e, \sigma \rangle \) are the eigenstates of the noninteracting Hamiltonian \( \hat{H}_0 \):

\[
\hat{H}_0 \mid n_{\omega}, m_e, \sigma \rangle = \omega \left( d \cdot \frac{N - 1}{2} + m_0 + m_e \right) \mid n_{\omega}, m_e, \sigma \rangle, \tag{31}\]

corresponding to \( m_e \) total excitation quanta, \( n_{\omega} \) of them carried by the center of mass. By the construction of the center-of-mass eigenstates [compare Eq. (29)] the total degeneracy of the noninteracting Hamiltonian, \( \Gamma^{\mu}_d (m_e) \), can be expressed as

\[
\Gamma^{\mu}_d (m_e) = \sum_{n_{\omega} = 0}^{m_e} g_d (n_{\omega}) \Gamma^{\mu}_d (m_e - n_{\omega}, 0), \tag{32}\]

where \( \Gamma^{\mu}_d (m_e, 0) \) is the degeneracy of the \( m_e \)-th excited level of a noninteracting system of \( N \) particles (bosons or fermions) without any excitation of the center of mass [obviously \( \Gamma^{\mu}_d (k, 0) = 0 \) if \( k < 0 \)]. \( \Gamma^{\mu}_d (m_e, 0) \) is also the number of independent solutions of equations (28). Equation (32) is the recurrence relation for \( \Gamma^{\mu}_d (m_e, 0) \) and its solutions are

\[
\Gamma^{\mu}_d (m_e, 0) = \sum_{k = 0}^{d} (-1)^k \left\lfloor \frac{d}{k} \right\rfloor \Gamma^{\mu}_d (m_e - k). \tag{33}\]

We are now ready to construct full solutions to the eigenvalue problem of the total Hamiltonian. The last step is to diagonalize the center of mass of the interacting Hamiltonian (that of frequency equal to \( 1 \)) in the basis of eigenstates of \( \hat{H}^{rel} \). First we note that

\[
[\hat{H}^{rel}, C_l] = 0, \tag{34}\]

which allows us to find such eigenstates of \( \hat{H}^{rel} \) which are annihilated by the center-of-mass destruction operators

\[
C_l \mid 0_{C}, m_e, \sigma \rangle = 0. \tag{35}\]

These states differ from the eigenstates of \( C_{\omega, l} \) operators by a simple squeezing transformation, therefore it is not difficult to find solution of the above equation:

\[
\mid 0_{C}, m_e, \sigma \rangle = N_C \sum_{n_{\omega}} \left( \frac{1 - \omega}{1 + \omega} \right)^{n_{\omega}} \times \prod_{l} \sqrt{\frac{(2n_{\omega, l} - 1)!!}{(2n_{\omega, l})!!}} \mid 2n_{\omega, m_e + 2n_{\omega}, \sigma} \rangle, \tag{36}\]

where \( N_C \) is the normalization factor. Because \( [\hat{H}, C_l^\dagger] = C_l^\dagger \), the operator \( C_l^\dagger \) generates excitation of the center of mass labeled by \( n_{C, l} \):

\[
\mid n_{C, m_e, \sigma} \rangle = \prod_{l} \frac{(C_l^\dagger)^{n_{C, l}}}{\sqrt{n_{C, l}!}} \mid 0_{C, m_e, \sigma} \rangle, \tag{37}\]

with \( n_c = \sum_{l} n_{C, l} \).

In this way we have obtained all eigenvectors \( \mid n_{C, m_e, \sigma} \rangle \) as well as eigenenergies of the total Hamiltonian, \( \hat{H} \), of the interacting system

\[
\hat{H} \mid n_{C, m_e, \sigma} \rangle = E_{m_e, n_{C}} \mid n_{C, m_e, \sigma} \rangle, \tag{38}\]

where the allowed energies are

\[
E_{m_e, n_{C}} = \omega \left( d \cdot \frac{N - 1}{2} + m_0 + m_e \right) + \frac{d}{2} + n_{C}. \tag{39}\]

All \( N \)-particle energy levels are, in general, multiple degenerated and the degeneracy factor, \( \Gamma^{\mu, int}_d (m_e, n_{C}) \), of the energy \( E_{m_e, n_{C}} \) can be expressed by the degeneracy of the noninteracting system:
\[
\Gamma_{d}^{\mu,\text{int}}(m_{e}, n_{e}) = g_{d}(n_{e}) \Gamma_{d}^{\mu}(m_{e}, 0). \tag{40}
\]

Thus we have shown that spectrum of the interacting Hamiltonian can be decomposed into two branches. The first one corresponds to the center-of-mass excitations and it is a single particle oscillatory spectrum of frequency equal to the external potential frequency. The second branch describes excitations of the relative degrees of freedom. The corresponding spectrum is that of harmonic oscillator of frequency \(\omega\) which depends on the interaction strength. The degeneracy factors of the interacting and noninteracting systems are directly related by formula (40).

V. CORRELATION FUNCTIONS AND STATISTICAL PROPERTIES

Here we want to apply our formalism to study some physically relevant quantities. The ground state for the boson system is given by a product of one-particle wave functions

\[
\psi_{B}(x_{1}, \ldots, x_{N}) = \mathcal{N}_{B} \exp \left[ -1/(2N)(1 - \omega) \left( \sum_{i=1}^{N} x_{i} \right)^{2} - \omega \sum_{i=1}^{N} x_{i}^{2}/2 \right]. \tag{41}
\]

Normalization factors have been denoted by \(\mathcal{N}_{B}\) and \(\mathcal{N}_{F}\). In order to characterize the ground state we will find one particle density function, two point correlation function, and density-density correlation. The first one gives spatial profiles of the trapped particles. Two point correlation function provides information on the coherence of the system. The density-density correlation, known also as the form-factor, plays an important role in interpretation of scattering experiments. The two-point correlation function is given by

\[
G(x, y) = \langle n_{C}, m_{e}, \sigma | \hat{\Psi}^{\dagger}(x) \hat{\Psi}(y) | n_{C}, m_{e}, \sigma \rangle. \tag{43}
\]

The particle density is \(n(x) = G(x, x)\). The density-density correlation function is defined as

\[
K(x, y) = \langle n_{C}, m_{e}, \sigma | \hat{\Psi}^{\dagger}(x) \hat{\Psi}(x) \hat{\Psi}^{\dagger}(y) \hat{\Psi}(y) | n_{C}, m_{e}, \sigma \rangle - G(x, x)G(y, y). \tag{44}
\]

We will give some examples for fermions as less known as compared to the boson case. The functions \(n(x)\), \(G(x,0)\) and \(K(x,0)\) for the ground state of 1D fermionic system of 50 particles are shown in Figs. 1 and 2. In Fig. 1(a) we see how the repulsive interactions change profile of density \(n\) in the harmonic trap. In Fig. 1(b) we see oscillations of the two-point correlation function that are due to the Fermi statistics. Figure 2 illustrates how the density-density correlation function \(K(x,0)\) changes due to the interatomic forces.

\[\text{FIG. 1. One-particle density function (a) two point correlation functions (b) for ground states of fermions in 1D for 50 particles with } \omega = 1, \text{ i.e., noninteracting system, dashed line; } \omega = 0.9, \text{ dotted line; and } \omega = 0.5, \text{ full line. All quantities are measured in harmonic oscillator units corresponding to frequency of the trap. Density is normalized to 1 and } G(x,0) \text{ is normalized accordingly.}\]

\[\text{Dashed line is printed for noninteracting system and shows that the correlation function for fermionic system decays rapidly with distance, whereas its range for the interacting system is comparable to the particle cloud size [compare Fig. 1(a)]. In contrast to this, } G(x,0) \text{ function exhibits qualitatively the same behavior for interacting and noninteracting systems.}\]

\[\text{FIG. 2. Two-particle density correlation function } K(x,0) \text{ in 1D for 50 particles and } \omega = 1, \text{ noninteracting system plotted with dashed line and } \omega = 0.9, \text{ full line. All quantities are measured in harmonic oscillator units corresponding to frequency of the trap. Function } K(x,0) \text{ is normalized to 1.}\]
The knowledge of all states allows one to find the thermodynamic properties of the system. In particular the entropy is related to the microcanonical partition function by the standard formula $S^m_d(E) = \log \{ \Gamma_d^m(E) \}$ where $E$ stands for one of the allowed energies, Eq. (39). Similarly the microcanonical temperature $1/T = \partial S^m_d(E)/\partial E$ may be easily calculated. For very large system, i.e. for large $N$ the contribution of the center-of-mass excitations (single particle spectrum) becomes negligible as compared to that coming from the relative $N-1$ degrees of freedom. Therefore thermodynamic characteristics of the interacting system are identical with quantities obtained for the ideal gas provided that the trap frequency is rescaled to the value $\omega$.

Although nothing prevents us to show the entropy for a large system, in Fig. 3 we plotted the entropy for $N=15$ particles in 3D in order to emphasize the center-of-mass contribution. The dashed lines show the entropy of noninteracting systems, the full lines describe contribution due to relative excitations only, while with dotted lines exhibit the total entropy of interacting systems (left three curves for bosons, right ones for fermions). It is clearly seen that for large energies all the curves (for a given particle statistics) have the same slope what indicates the same thermodynamic properties. For large particle number, say $(N>100)$, the curves are practically indistinguishable. In that limit the contribution from the center-of-mass degrees of freedom becomes negligible and we recover the expected results for noninteracting particles with properly rescaled trap frequency $\omega$.

Thermodynamics of noninteracting Bose gas trapped by harmonic potential was studied, e.g., in Ref. [9].

We will now calculate the grand canonical partition function for our system. Since our interaction has a long range character the effective interaction strength grows with the number of particles. This is obviously not the case for more realistic short range interactions. However we can go around this problem allowing the frequency $\Omega$ be $N$ dependent in such a way that the effective interaction does not depend on the total particle number. Thermodynamic concepts are meaningful for such a system, as opposed to systems with nonrenormalized long range interactions.

The crucial part is the calculation of the grand canonical partition function which, in our case, is

$$Z(\beta, z) = \sum_{N=0}^{\infty} \sum_{n_C, n_e} \Gamma_d^{\mu, \text{int}}(m_e, n_C) e^{-\beta E_{n_e, n_C}}, \quad (45)$$

with $\beta$ is inverse temperature in dimensionless units. Using the fact that the microcanonical partition function is a product of single particle one (corresponding to $d$-dimensional harmonic oscillator) and microcanonical partition function of $N-1$ oscillatory degrees of freedom (relative excitations), we can express the grand canonical partition function with the help of the corresponding function of the noninteracting system of particles trapped in the harmonic potential of frequency $\omega$:

$$Z(\beta, z) = \left( \frac{1 - e^{-\beta \omega}}{1 - e^{-\beta}} \right)^d Z_0^\mu(\omega), \quad (46)$$

where $Z_0^\mu$ is the grand partition function for the noninteracting particles of a given symmetry; $\mu = B$ for bosons and $\mu = F$ for fermions. The prefactor describing the center of mass excitations does not influence significantly the thermodynamic properties of the system. Thus we have related the thermodynamics of the interacting system to that of independent particles. The main difference lies in the effective frequency $\omega$. Therefore all results obtained for independent particles, such as the Bose-Einstein phase transition, remain valid for the interacting particles.

VI. CONCLUSIONS

We have performed a detailed analysis of a system of quantum particles trapped by external harmonic potential and interacting via harmonic attractive as well as repulsive forces. The spectrum, i.e., energy levels and their degeneracies, is well known for identical particles (fermions and bosons) has been found. Although not necessarily realistic, this specific form of a two-body interaction enables us to find analytical solutions. As the model happened to be analytically solvable our results can be used to test various approximations that are indispensable in the treatment of more realistic situations. In our model we were able to reduce, in a sense, the problem to that of a noninteracting Hamiltonian, which is the first necessary step towards the solution of the problem, is relatively easy—it is
enough to change variables. However, due to the quantum statistics that enforces the proper symmetrization of the eigenfunctions (in the original variables), the abovementioned reduction to noninteracting case is far from being trivial. The crucial point is the observation that it is always possible to introduce properly rescaled noninteracting Hamiltonian having the desirable property. Namely, in new variables (that diagonalize the total interacting Hamiltonian) it separates into two independent parts, one of which is identical with this part of diagonalized interacting Hamiltonian that describes internal degrees of freedom. Implementing such a procedure we have shown that the mutual interactions modify significantly the spectrum of the system. Energy spectrum splits into two branches. One of them stays the same as for the noninteracting system and describes the center-of-mass excitations. Degeneracy of the center-of-mass energy is the same as the degeneracy of one particle in $d$-dimensional harmonic potential. Energy levels in the second branch of the energy spectrum are separated by a new, changed by interaction strength, energy scale $\omega$. Their degeneracies, although high, are reduced in comparison to the rescaled noninteracting problem and they grow with energy in a similar way as the degeneracy of the noninteracting system. Therefore the interacting system has the same thermodynamical properties with properly rescaled energy values.

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