Full quantum dynamical description of the dissipative Bose-Hubbard model

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*PRX Quantum **2**, 010319 (2021)*
*Quantum **5**, 455 (2021)*

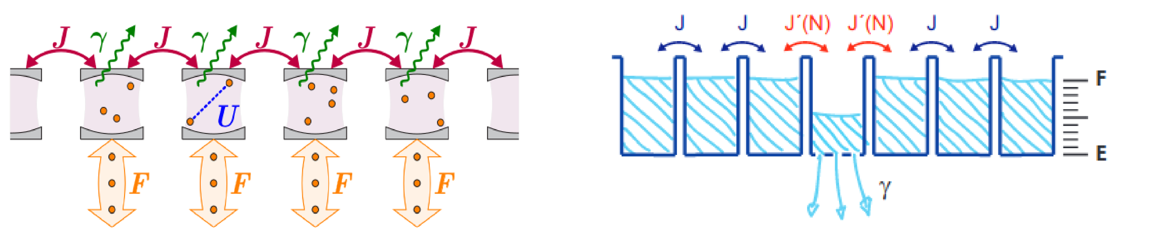
Conclusion: full quantum dynamics of up to millions of sites can be done in the right parameter ranges

Dissipative Bose-Hubbard model

$$\widehat{H} = \sum_j \widehat{H}_j - \sum_{\text{connections } i,j} \left[J_{ij}\widehat{a}_j^\dagger\widehat{a}_i + J_{ij}^*\widehat{a}_i^\dagger\widehat{a}_j\right]$$

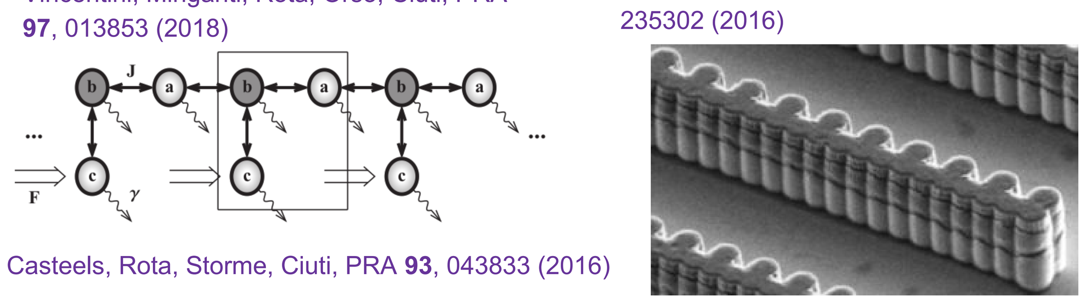
$$\widehat{H}_j = -\Delta_j\widehat{a}_j^\dagger\widehat{a}_j + \frac{U_j}{2}\widehat{a}_j^\dagger\widehat{a}_j^\dagger\widehat{a}_j\widehat{a}_j + F_j\widehat{a}_j^\dagger + F_j^*\widehat{a}_j$$

$$\frac{\partial\widehat{\rho}}{\partial t} = -i\left[\widehat{H},\widehat{\rho}\right] + \sum_j \frac{\gamma_j}{2}\left[2\widehat{a}_j\widehat{\rho}\widehat{a}_j^\dagger - \widehat{a}_j^\dagger\widehat{a}_j\widehat{\rho} - \widehat{\rho}\widehat{a}_j^\dagger\widehat{a}_j\right]$$



Vincentini, Minganti, Rota, Orso, Ciuti, PRA **97**, 013853 (2018)

Labouvie, Santra, Heun, Ott, PRL **116**, 235302 (2016)



Casteels, Rota, Storme, Ciuti, PRA **93**, 043833 (2016)

Baboux, Ge, Jacqmin, Biondi, Galopin, Lemaitre, Le Gratiet, Sagnes, Schmidt, Tureci, Amo, Bloch, PRL **116**, 066402 (2016)

Positive-P representation

M subsystems (modes, sites, volumes) labeled by j

Coherent state basis, complex, *local* α_j $\quad |\alpha_j\rangle_j = e^{-|\alpha_j|^2/2}\, e^{\alpha_j\widehat{a}_j^\dagger}|\text{vac}\rangle$

Local operator kernel $\quad \widehat{\Lambda}(\boldsymbol{\lambda}) = \bigotimes_j \frac{|\alpha_j\rangle_j\langle\beta_j^*|_j}{\langle\beta_j^*|_j|\alpha_j\rangle_j}$

"ket" amplitude α_j
"bra" amplitude β_j^*

full system configuration $\quad \boldsymbol{\lambda} = \{\alpha_1,\dots,\alpha_M,\beta_1,\dots,\beta_M\}$

Full density matrix $\quad \widehat{\rho} = \int d^{4M}\boldsymbol{\lambda}\, P_+(\boldsymbol{\lambda})\,\widehat{\Lambda}(\boldsymbol{\lambda})$

Correlations between subsystems are all in the distribution of configurations

$P_+(\boldsymbol{\lambda})$ The distribution is positive, real ⟶ let's SAMPLE IT !

Quantum dynamics (1-mode example):

Density matrix $\widehat{\rho}$ ↔ distribution P_+ for the fields ↔ random samples of the fields α β

Master equation: $\widehat{H} = \frac{U}{2}\widehat{a}^\dagger\widehat{a}^\dagger\widehat{a}\widehat{a} - \Delta\widehat{a}^\dagger\widehat{a}$ $\quad \hbar = 1$

$$\frac{\partial\widehat{\rho}}{\partial t} = -i\left[\widehat{H},\widehat{\rho}\right] + \frac{\gamma}{2}\left(2\widehat{a}\widehat{\rho}\widehat{a}^\dagger - \widehat{a}^\dagger\widehat{a}\widehat{\rho} - \widehat{\rho}\widehat{a}^\dagger\widehat{a}\right)$$

dissipation γ

apply identities

Fokker Planck equation

$$\frac{\partial P_+}{\partial t} = \left\{-\frac{\partial}{\partial\alpha}\left(-iU\alpha\beta + i\Delta - \frac{\gamma}{2}\right)\alpha - \frac{\partial}{\partial\beta}\left(iU\alpha\beta - i\Delta - \frac{\gamma}{2}\right)\beta + \frac{\partial^2}{\partial\alpha^2}\left(\frac{-iU}{2}\right)\alpha^2 + \frac{\partial^2}{\partial\beta^2}\left(\frac{iU}{2}\right)\beta^2\right\}P_+$$

deterministic (ket) — deterministic (bra) — quantum noise

Stochastic (Langevin) equations: *stochastic correspondence*

$$\frac{d\alpha}{dt} = \left(-iU\alpha\beta + i\Delta - \frac{\gamma}{2}\right)\alpha + \sqrt{-iU}\,\alpha\,\xi(t)$$

$$\frac{d\beta}{dt} = \left(+iU\alpha\beta - i\Delta - \frac{\gamma}{2}\right)\beta + \sqrt{+iU}\,\beta\,\tilde{\xi}(t)$$

mean field part — quantum noise part — different noises

White noise $\langle\xi(t)\xi(t')\rangle_{\text{stoch}} = \delta(t-t')$

Dissipative Bose-Hubbard dynamics:

$$\frac{\partial\alpha_j}{\partial t} = i\Delta_j\alpha_j - iU_j\alpha_j^2\tilde{\alpha}_j^* - iF_j - \frac{\gamma_j}{2}\alpha_j + \sqrt{-iU_j}\,\alpha_j\,\xi_j(t) + \sum_k iJ_{kj}\alpha_k,$$

$$\frac{\partial\tilde{\alpha}_j}{\partial t} = i\Delta_j\tilde{\alpha}_j - iU_j\tilde{\alpha}_j^2\alpha_j^* - iF_j - \frac{\gamma_j}{2}\tilde{\alpha}_j + \sqrt{-iU_j}\,\tilde{\alpha}_j\,\tilde{\xi}_j(t) + \sum_k iJ_{kj}\tilde{\alpha}_k$$

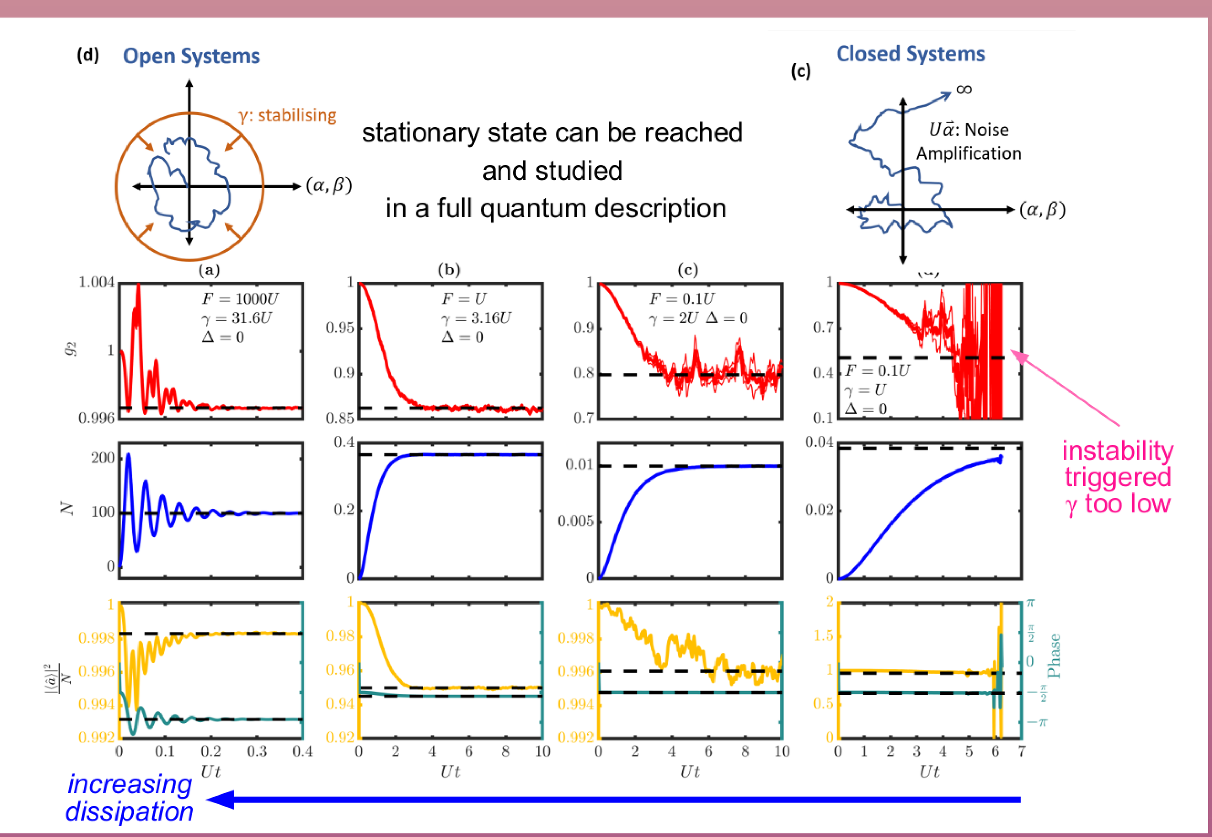
White Gaussian noise deals with interparticle collisions

$$\langle\xi_j(t)\xi_k(t')\rangle_s = \delta(t-t')\delta_{jk}, \quad \langle\tilde{\xi}_j(t)\tilde{\xi}_k(t')\rangle_s = \delta(t-t')\delta_{jk}$$

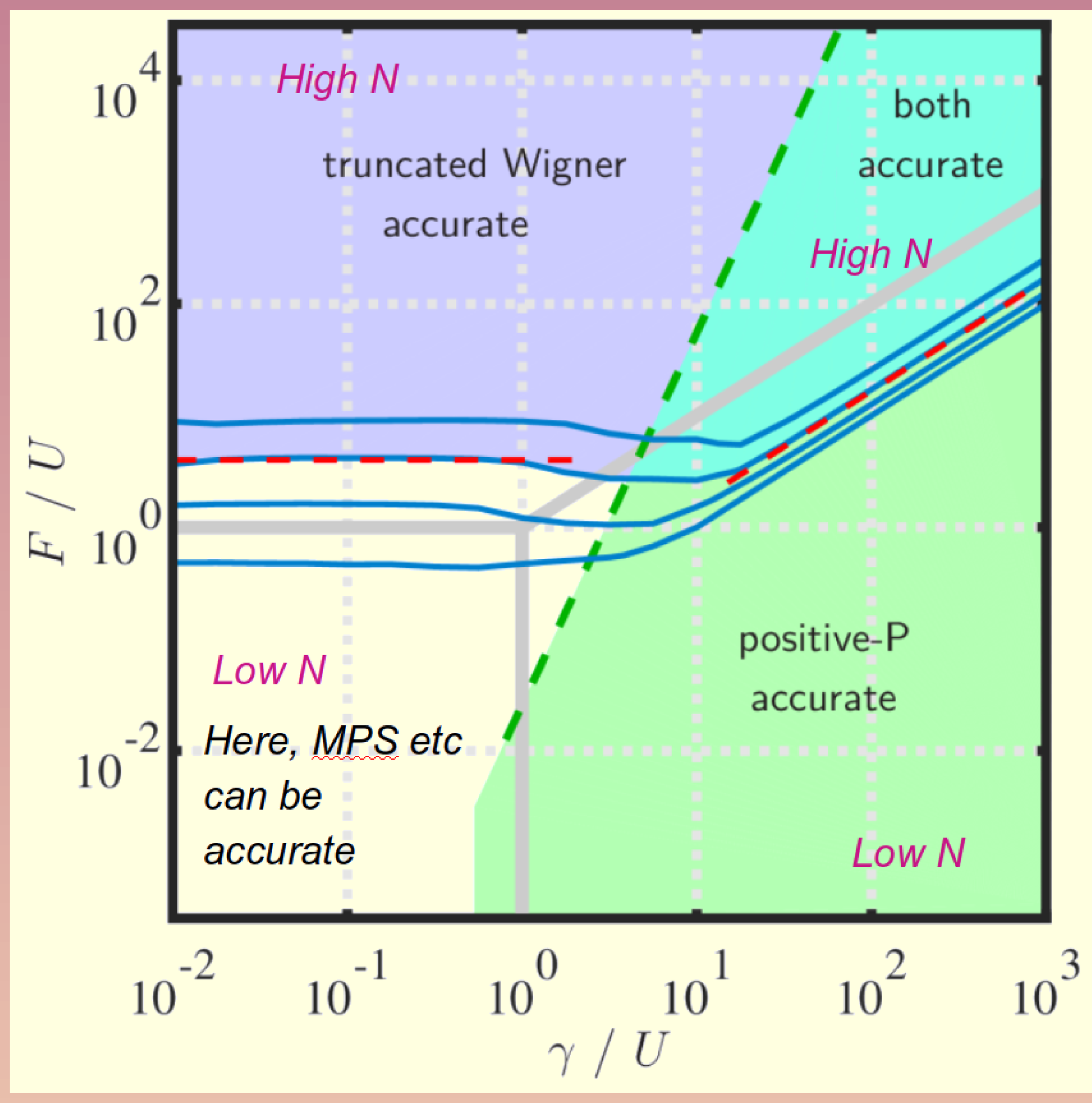
The rest of the equations is basically mean field

Stabilisation by dissipation – 1 mode

stationary state can be reached and studied in a full quantum description



REGIONS OF APPLICABILITY



Positive-P stability:

determined by single site parameters

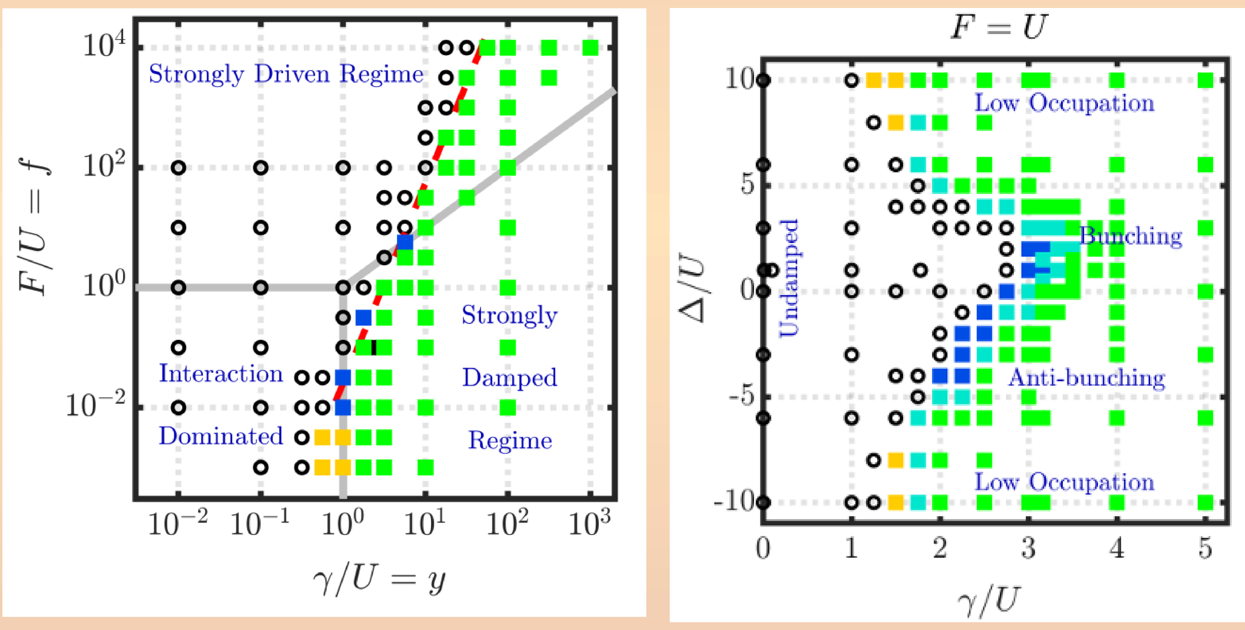
$\gamma \gtrsim 3U\left(\frac{F}{U}\right)^{0.30}$

$N \sim \left(\frac{F}{U}\right)^{2/3}$

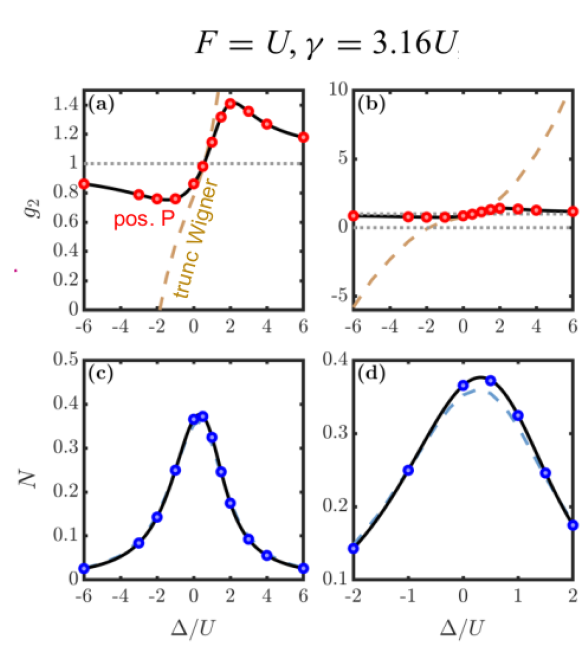
resultant occupation dependence of stability region

$\gamma \gtrsim 3U\sqrt{N}$

Single mode testing:



truncated Wigner (in)accuracy



Accuracy indicator:

$$\Delta_{\text{TW}} = \max_j\left[\Delta_{\text{sys}}^{(j)}, \Delta_{\text{stat}}^{(j)}\right]$$

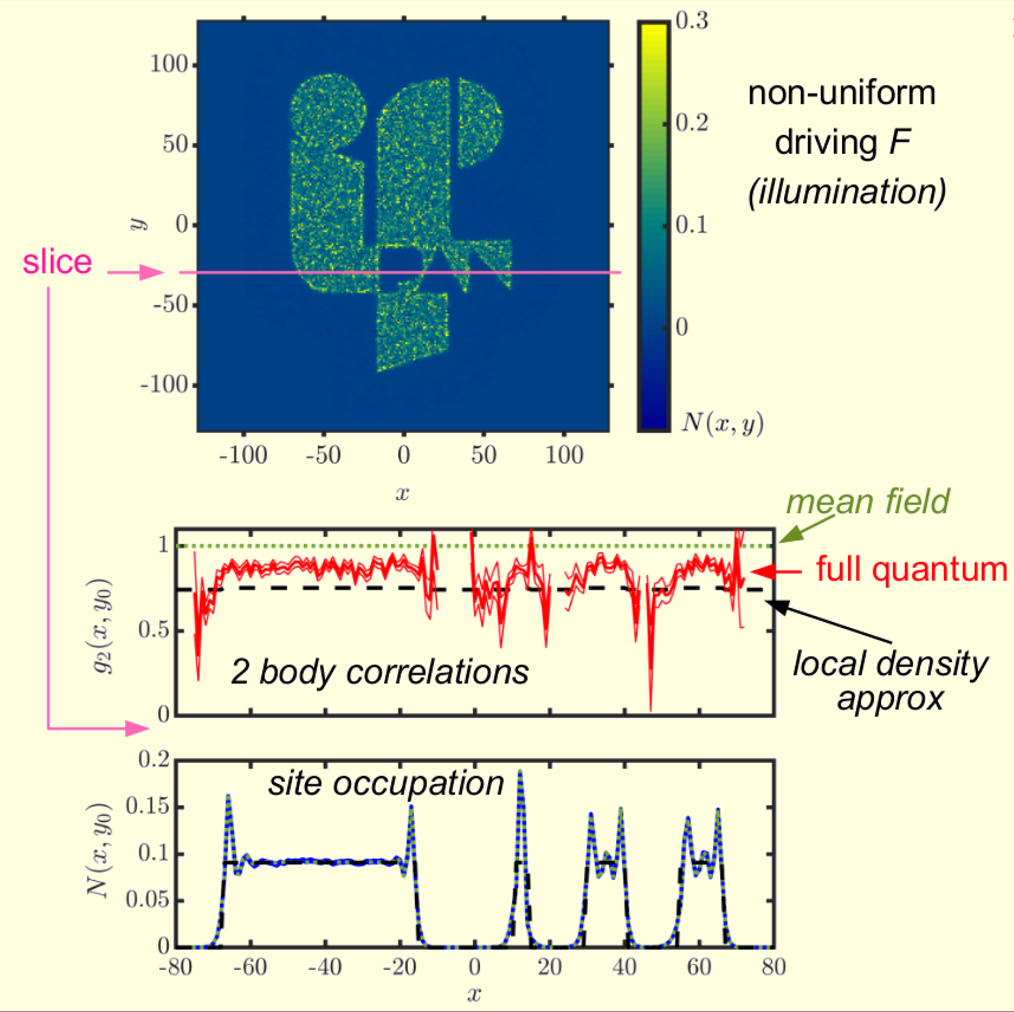
$$\Delta_{\text{sys}}^{(j)} = \left|\frac{O^{(j)} - O_{\text{ex}}^{(j)}}{O_{\text{ex}}^{(j)}}\right|, \qquad \Delta_{\text{stat}}^{(j)} = \frac{\delta_{\text{stat}}O^{(j)}}{|O^{(j)}|}.$$

Using observables:

$$N = \langle|\alpha|^2\rangle_s - \tfrac{1}{2}$$

$$g_2 = \frac{\langle|\alpha|^4 - 2|\alpha|^2 + 1/2\rangle_s}{N^2}$$

Large nonuniform system 256 x 256 sites



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Multi-time correlations

Expressed in terms of Heisenberg operators

$$\widehat{A}(t) = e^{i(t-t_0)\widehat{H}/\hbar}\,\widehat{A}(t_0)\,e^{-i(t-t_0)\widehat{H}/\hbar}$$

Time ordered correlation functions.
Correspond to all sequences of measurements

$$\langle\widehat{A}_1(t_1)\widehat{A}_2(t_2)\cdots\widehat{A}_N(t_N)\widehat{B}_1(s_1)\widehat{B}_2(s_2)\cdots\widehat{B}_M(s_M)\rangle \qquad t_1 \leqslant t_2 \leqslant \ldots \leqslant t_N, \quad s_1 \geqslant s_2 \geqslant \ldots \geqslant s_M$$

time grows ⟶ ⟵ time grows

Heisenberg equations of motion:

$$\frac{d\,\widehat{a}(t)}{dt} = \left(-iU\widehat{a}^\dagger(t)\widehat{a}(t) + i\Delta - \frac{\gamma}{2}\right)\widehat{a}(t)$$

$$\frac{d\,\widehat{a}^\dagger(t)}{dt} = \widehat{a}^\dagger(t)\left(+iU\widehat{a}^\dagger(t)\widehat{a}(t) - i\Delta - \frac{\gamma}{2}\right)$$

positive-P equations of motion

$$\frac{d\alpha}{dt} = \left(-iU\alpha\beta + i\Delta - \frac{\gamma}{2}\right)\alpha + \sqrt{-iU}\alpha\xi(t)$$

$$\frac{d\beta}{dt} = \left(+iU\alpha\beta - i\Delta - \frac{\gamma}{2}\right)\beta + \sqrt{+iU}\beta\tilde{\xi}(t)$$

Notice similarity

Main result

Normal ordering: positive-P variables

$$\langle\widehat{a}_{p_1}^\dagger(t_1)\cdots\widehat{a}_{p_N}^\dagger(t_N)\widehat{a}_{q_1}(s_1)\cdots\widehat{a}_{q_M}(s_M)\rangle = \langle\beta_{p_1}(t_1)\cdots\beta_{p_N}(t_N)\alpha_{q_1}(s_1)\cdots\alpha_{q_M}(s_M)\rangle_{\text{stoch}}$$

Anti-normal ordering: Q distribution variables

$$\langle\widehat{a}_{p_1}(t_1)\cdots\widehat{a}_{p_N}(t_N)\widehat{a}_{q_1}^\dagger(s_1)\cdots\widehat{a}_{q_M}^\dagger(s_M)\rangle = \langle\alpha'_{p_1}(t_1)\cdots\alpha'_{p_N}(t_N)\beta'_{q_1}(s_1)\cdots\beta'_{q_M}(s_M)\rangle_{\text{stoch}}$$

conversion P ⟶ Q

$$\alpha'_j = \alpha_j + \zeta_j \quad ; \quad \beta'_j = \beta_j + \zeta_j^*$$

$$\langle\zeta_j\rangle_{\text{stoch}} = 0\;;\;\langle\zeta_j\zeta_k\rangle_{\text{stoch}} = 0\;;\;\langle\zeta_j^*\zeta_k\rangle_{\text{stoch}} = 1$$

Mixed ordering:
1) sample what is possible using positive-P variables
2) convert variables to doubled-Q
3) sample what is possible using Q variables

Coverage

Order (number of operators)	2nd order	3rd order	4th order
Total permutations	**12**	**56**	**240**
single time correlations	4	8	16
multi-time accessible with P representation	4	14	36
additional accessible with Q representation	4	14	36
additional accessible with mixed order (Sec. 5.4)	–	12	72
Total doable	**12**	**48**	**160**
time ordered not doable	–	–	–
Not time ordered, not doable	–	8	80

Table 2: A tally of $\widehat{a}$, $\widehat{a}^\dagger$ products involving up to four operators, evaluated at one of two times. The general form considered is $\langle\widehat{A}(t_a)\widehat{B}(t_b)\widehat{C}(t_c)\widehat{D}(t_d)\rangle$, where $\widehat{A},\widehat{B},\widehat{C},\widehat{D}$ can be either of $\widehat{a}$ or $\widehat{a}^\dagger$ (same mode), and the time arguments can take up to two distinct times $t=0$ and $t=\tau>0$.

Order (number of operators)	1st order	2nd order	3rd order
Total permutations	**2**	**12**	**104**
single time correlations	2	4	8
multi-time accessible with P representation	–	4	22
additional accessible with Q representation	–	4	22
additional accessible with mixed order (Sec. 5.4)	–	–	22
Total doable	**2**	**12**	**74**
time ordered not doable	–	–	6
Not time ordered, not doable	–	–	24

Table 1: A tally of $\widehat{a}$, $\widehat{a}^\dagger$ product permutations that can/cannot be evaluated with the various approaches discussed. The general form considered is $\langle\widehat{A}(t_a)\widehat{B}(t_b)\widehat{C}(t_c)\rangle$, where $\widehat{A},\widehat{B},\widehat{C}$ can be either of $\widehat{a}$ or $\widehat{a}^\dagger$ (same mode), and the time arguments can take up to three distinct times $t_1 < t_2 < t_3$. Permutations with the same time topology (e.g. $\widehat{A}(t_1)\widehat{B}(t_1)\widehat{C}(t_2)$ and $\widehat{A}(t_2)\widehat{B}(t_2)\widehat{C}(t_3)$) are counted only once.

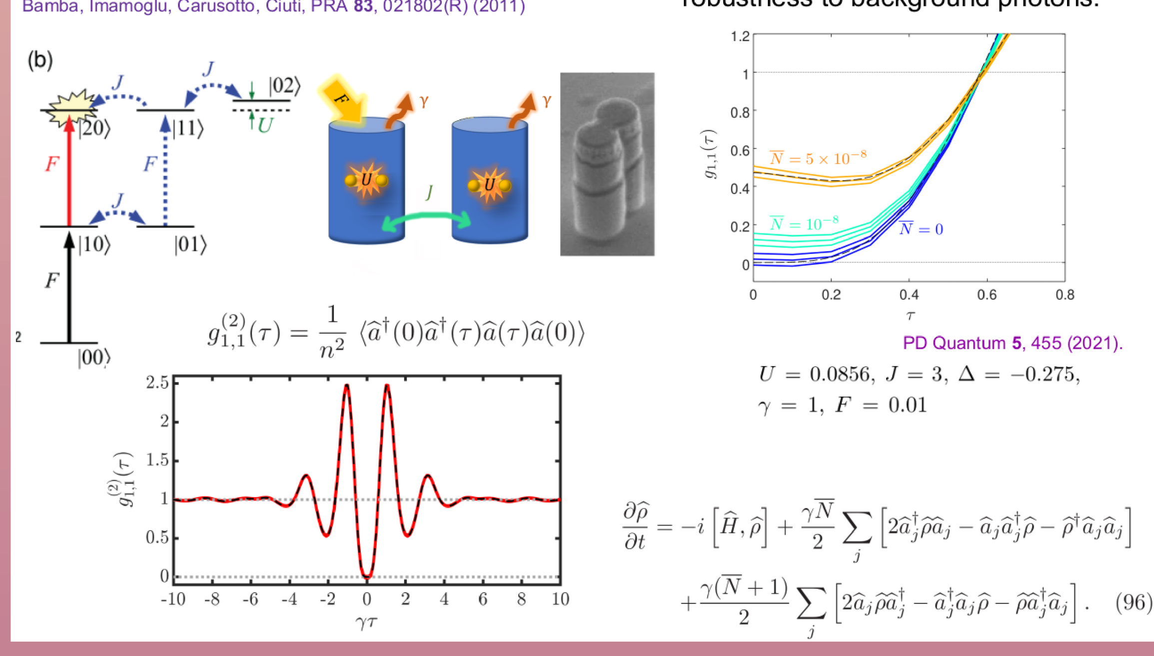
Unconventional photon blockade

Complete antibunching,
Subtle interference effect $\quad U \ll \gamma$

potential single-photon source

Liew, Savona, PRL **104**, 183601 (2010)
Bamba, Imamoglu, Carusotto, Ciuti, PRA **83**, 021802(R) (2011)

robustness to background photons:



PD Quantum **5**, 455 (2021).

$$g_{1,1}^{(2)}(\tau) = \frac{1}{n^2}\langle\widehat{a}^\dagger(0)\widehat{a}^\dagger(\tau)\widehat{a}(\tau)\widehat{a}(0)\rangle$$

$U = 0.0856,\ J = 3,\ \Delta = -0.275,\ \gamma = 1,\ F = 0.01$

$$\frac{\partial\widehat{\rho}}{\partial t} = -i\left[\widehat{H},\widehat{\rho}\right] + \frac{\gamma\overline{N}}{2}\sum_j\left[2\widehat{a}_j^\dagger\widehat{\rho}\widehat{a}_j - \widehat{a}_j\widehat{a}_j^\dagger\widehat{\rho} - \widehat{\rho}\widehat{a}_j\widehat{a}_j^\dagger\right] + \frac{\gamma(\overline{N}+1)}{2}\sum_j\left[2\widehat{a}_j\widehat{\rho}\widehat{a}_j^\dagger - \widehat{a}_j^\dagger\widehat{a}_j\widehat{\rho} - \widehat{\rho}\widehat{a}_j^\dagger\widehat{a}_j\right]. \quad (96)$$

Have a dissipative system you want to simulate?

non-uniform ?
time-dependent ??

Contact us ;-)