Properties of quantum graphs and microwave networks split at edges and vertices

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Abstract

We analyze the situation when the original graph is split at edges and vertices into two disconnected subgraphs. We show that there is a relationship between the generalized Euler characteristic $\mathcal{E}_o(|V_{D_o}|)$ of the original graph and the generalized Euler characteristics $\mathcal{E}_i(|V_{D_i}|)$, $i = 1, 2$, of two disconnected subgraphs, where $|V_{D_o}|$ and $|V_{D_i}|$, $i = 1, 2$, are the numbers of vertices with the Dirichlet boundary conditions in the graphs [1-4]. Theoretical predictions are verified experimentally using microwave networks which simulate quantum graphs.

Introduction

Quantum graphs were introduced by Linus Pauling and recently they have been used to model a large variety of physical systems and phenomena such as quantum wires, mesoscopic quantum systems, multi-walker and percolated quantum walks, and the photon number statistics of coherent light.

- The formula for calculating $\mathcal{E}(|V_D|)$ for graphs and networks:

$$\mathcal{E}(|V_D|) := \chi - |V_D| = 8\pi^2 \sum_{k_n \in \Sigma(L^{\mathrm{N,D}}(\Gamma))} \frac{\sin(k_n/t)}{(k_n/t)((2\pi)^2 - (k_n/t)^2)}\Big|_{t \geqslant t_0}$$

- $\sum (L^{\mathrm{N,D}}(\Gamma))$ – Spectrum of the Laplacian $L^{\mathrm{N,D}}(\Gamma)$ with standard vertex conditions.
- k_n – Square roots of the Eigen energies λ_n.
- $t = t_0 = \frac{1}{2l_{min}}$

- The formula for the minimum number of the lowest eigen--values required to calculate $\mathcal{E}(|V_D|)$ with ϵ accuracy:

$$K \geqslant |V| + 2\mathcal{L}t\left[1 - \exp\left(\frac{-\epsilon\pi}{\mathcal{L}t}\right)\right]^{-1/2}$$

- $|V|$ – total number of graph vertices.
- $\mathrm{L} = \sum_{e \in E} l_e$ – total length of the graph.
- ϵ – accuracy of determining $\mathcal{E}$.

Experimental setup and results

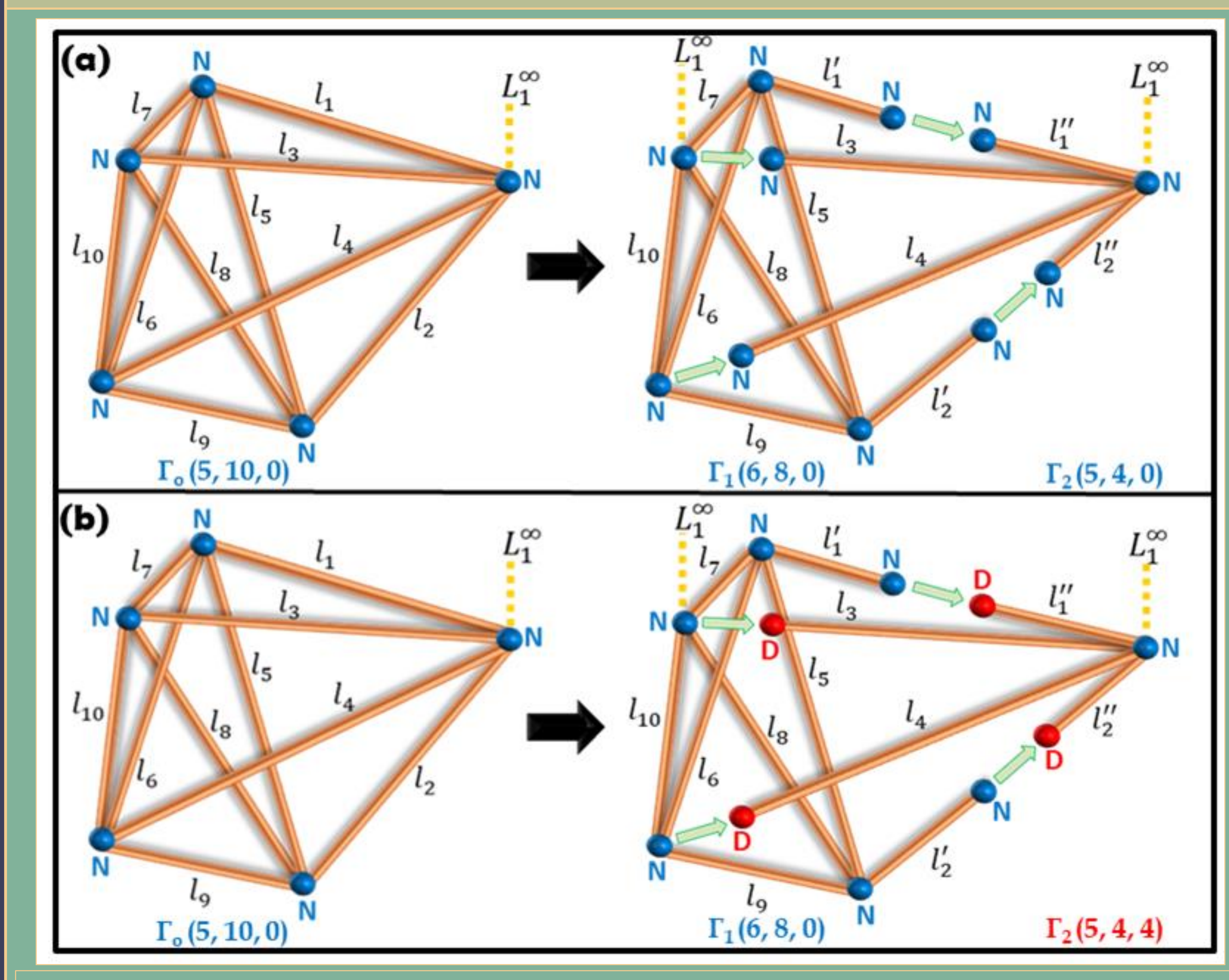

Figure 1. (a) The scheme of the original network $\Gamma_o(5, 10, 0)$ which was split at two edges and two vertices into two subnetworks $\Gamma_1(6, 8, 0)$ and $\Gamma_2(5, 4, 0)$. In this case all networks possess vertices with the Neumann boundary conditions which are marked by blue dots and capital letters *N*. (b) The original network $\Gamma_o(5, 10, 0)$ was split at two edges and two vertices into two subnetworks $\Gamma_1(6, 8, 0)$ and $\Gamma_2(5, 4, 4)$. The subnetwork $\Gamma_2(5, 4, 4)$ possesses the mixed boundary conditions: four vertices with the Dirichlet boundary conditions (red dots and capital letters *D*) and one vertex with the Neumann boundary condition (blue dot and the capital letter *N*). marks the semi-infinite lead attached to the networks.

- Neumann boundary condition

$$N_{\sigma_{e,e'}}(v) = \frac{2}{d_v} - \delta_{e,e'}$$

d_v – **degree of the vertex 'v'**
$\delta_{e,e'}$ – **Kronecker delta**

- Dirichlet boundary condition

$$D_{\sigma_{e,e'}}(v) = -\delta_{e,e'}$$

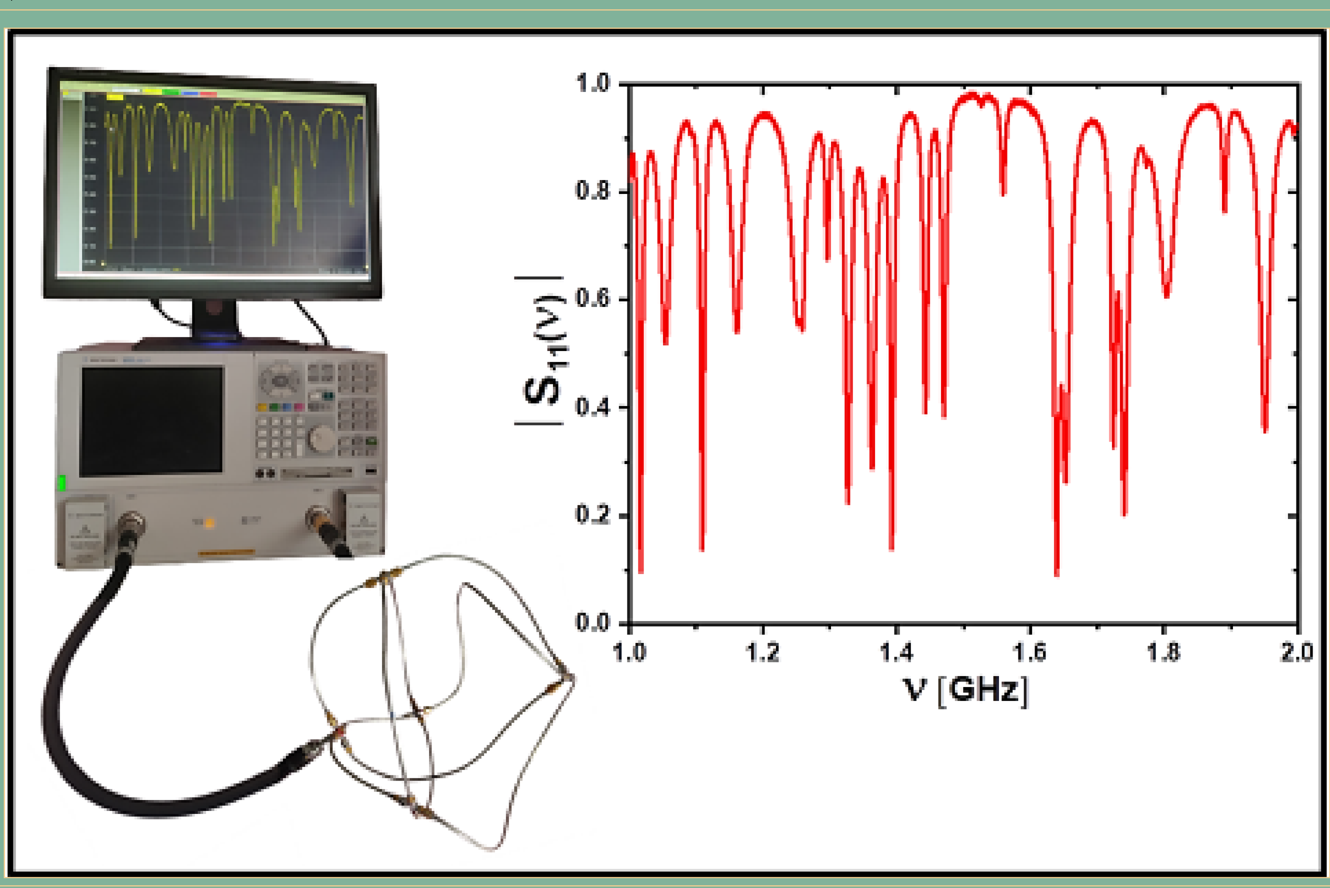

Figure 2. The experimental set-up for measuring the resonances of the microwave networks. It contains Agilent E8364B vector network analyzer (VNA) and HP 85133-60016 flexible microwave cable that connects the VNA to the network. The microwave network $\Gamma_o(5, 10, 0)$ possesses five vertices with the Neumann boundary conditions. The inset shows the spectrum of this network in the frequency range $\nu = [1, 2]$ GHz.

- **The generalized Euler characteristics evaluated for the networks with the Neumann and mixed boundary conditions as a function of the parameter t.**

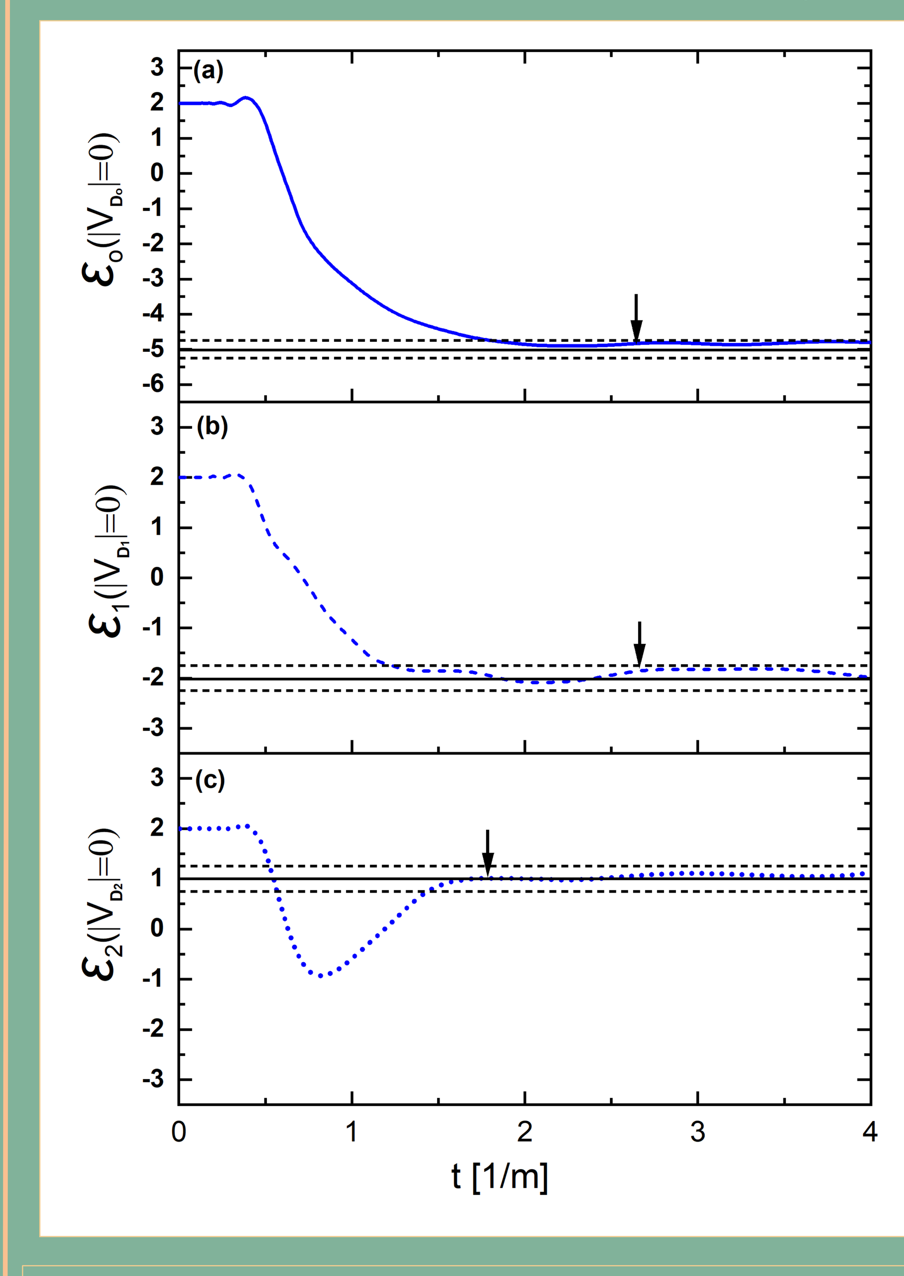

Figure 3. Panels (a), (b), and (c) show the generalized Euler characteristics (blue full line), (blue dashed line), and (blue dotted line) of the networks $\Gamma_o(5, 10, 0)$, $\Gamma_1(6, 8, 0)$, and $\Gamma_2(5, 4, 0)$, respectively.

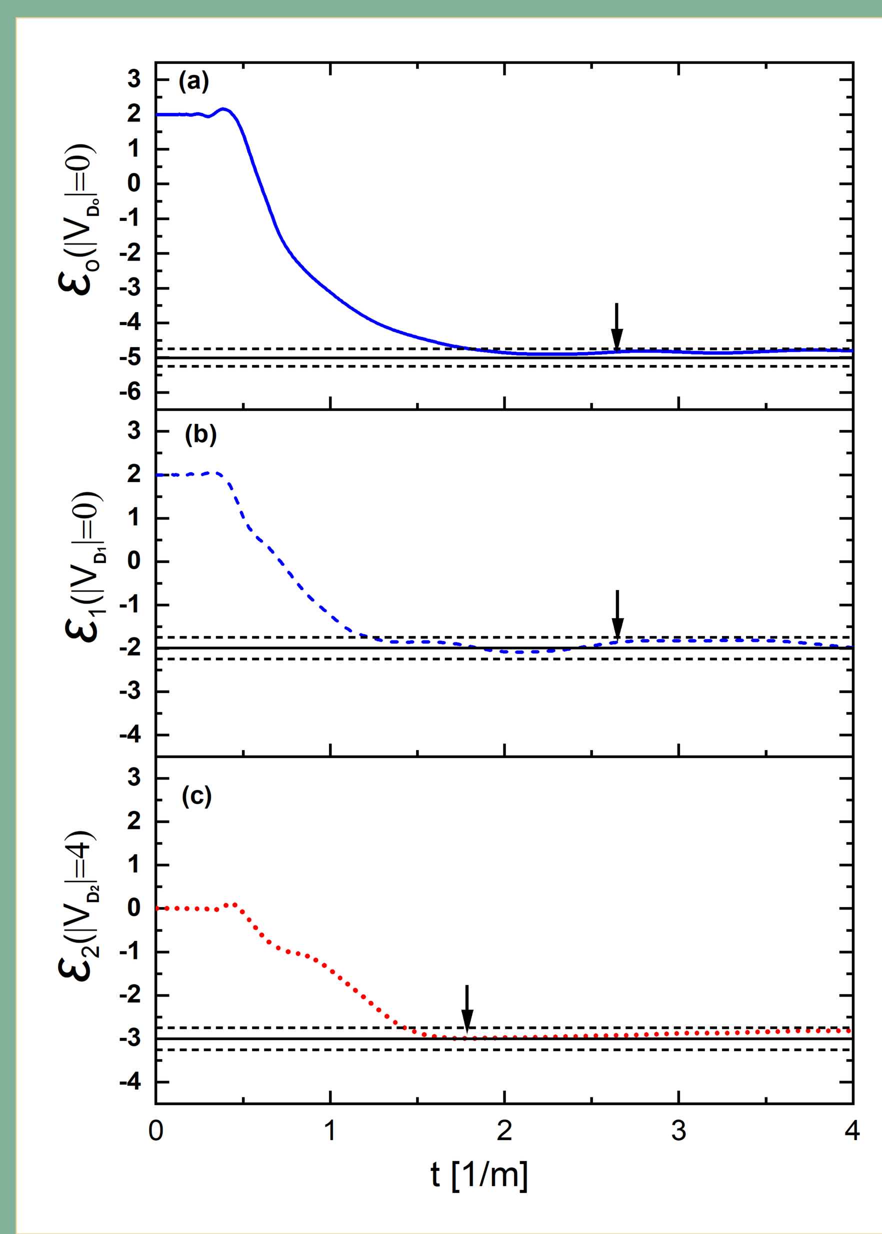

Figure 4. Panels (a), (b), and (c) show the generalized Euler characteristics (blue full line), (blue dashed line), and (blue dotted line) of the networks $\Gamma_o(5, 10, 0)$, $\Gamma_1(6, 8, 0)$, and $\Gamma_2(5, 4, 4)$, respectively.

In all cases, the plateaux at the generalized Euler characteristics start close to the points $t_{0_o} = 2.66\ \mathrm{m}^{-1}$, $t_{0_1} = 2.66\ \mathrm{m}^{-1}$, and $t_{0_2} = 1.78\ \mathrm{m}^{-1}$, respectively. The black broken lines show the limits of the assumed accuracy $\mathcal{E}_q(|V_{D_q}|) \pm 1/4$, where q = o, 1, and 2.

	Total Length (m)	l_{min} (m)	K_{min}	Frequency range (GHz)	$\mathcal{E}(\lvert V_D \rvert)$
Original Network	3.821 ± 0.005	$l_7 = 0.188$	80	[0.010, 3.18]	-5 ± 1/4
Subnetwork 1	2.444 ± 0.005	$l_7 = 0.188$	45	[0.010, 2.80]	-2 ± 1/4
Subnetwork 2 with Neumann boundary condition	1.375 ± 0.005	$l_2 = 0.281$	15	[0.010, 1.60]	1 ± 1/4
Subnetwork 2 with mix boundary condition	1.375 ± 0.005	$l_2 = 0.281$	15	[0.010, 1.60]	-3 ± 1/4

Summary and Conclusions

We derived the relationship which links the generalized Euler characteristic $\mathcal{E}_o|V_{D_o}|)$ of the original graph (network) which was split at edges and vertices into two subgraphs (subnetworks) with their generalized Euler characteristics $\mathcal{E}_i(|V_{D_i}|)$, $i = 1, 2$. We demonstrated that the experimental evaluation of the generalized Euler characteristics $\mathcal{E}_o|V_{D_o}|)$ and $\mathcal{E}_i(|V_{D_i}|)$, $i = 1, 2$, also in the case of the mixed boundary conditions allow to determine the sum of the edges and vertices $|E_c| + |V_c|$ where the two subgraphs (subnetworks) were initially connected. The presented method appeared to be especially useful for the graphs (networks) with the Neumann boundary conditions because it allows to find $|E_c| + |V_c|$ without knowing graphs (networks) topologies. In this case if we additionally know either $|E_c|$ or $|V_c|$, the generalized Euler characteristic allows us to find accordingly the exact number of vertices $|V_c|$ or edges $|E_c|$ where the original graph (network) was split.

References

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