Division of Theoretical Physics

NOISE REPRESENTATIONS OF OPEN SYSTEM DYNAMICS

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I. INTRODUCTION

Consider the dynamics of the system (*S*) open to its environment (*E*),

$$\hat{\rho}_S(t) = \mathrm{tr}_E\left(e^{-it\hat{H}_{SE}}\hat{\rho}_S\otimes\hat{\rho}_E e^{it\hat{H}_{SE}}\right)$$

where the total Hamiltonian of the system--environment complex is given by,

$$\hat{H}_{SE} = \hat{H}_S + \hat{S}\otimes\hat{E} + \hat{H}_E$$

It is known that for certain system--environment complexes one can represent the dynamics of *S*, as if it was driven by an external noise, or the **surrogate field**, that replaces the environmental degrees of freedom and simulates their influence on the open system

$$\mathrm{tr}_E\left(e^{-it\hat{H}_{SE}}\hat{\rho}_S\otimes\hat{\rho}_E e^{it\hat{H}_{SE}}\right) = \overline{\hat{U}(t|\xi)\hat{\rho}_S\hat{U}^\dagger(t|\xi)}$$

where the bar indicated the average over stochastic elements of the surrogate field ξ and the per-trajectory unitary evolution operator reads

$$\hat{U}(t|\xi) = \mathcal{T}e^{-i\int_0^t(\hat{H}_S+\xi(\tau)\hat{S})d\tau}$$

Problem to solve: How and why is it possible that a composite quantum system admits such a surrogate field representation?

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II. QUASI-PROBABILITY REPRESENTATION

The system state can be expressed in terms of average-like "trajectory integral"

$$\mathrm{tr}_E\left(e^{-it\hat{H}_{SE}}\hat{\rho}_S\otimes\hat{\rho}_E e^{it\hat{H}_{SE}}\right) = \int\mathcal{D}\xi(t)\mathcal{D}\zeta(t)\mathcal{Q}[\xi(t),\zeta(t)]\mathcal{U}(t|\xi,\zeta)\hat{\rho}_S$$

where the trajectories ξ and ζ path through domain defined by the eigenvalues of the environment-side coupling operator

$$\hat{E} = \sum_n e_n|n\rangle\langle n| = \sum_\xi \xi\sum_{n:e_n=\xi}|n\rangle\langle n|$$

the super-operator acting on the initial state has the property $\mathcal{U}(t|\xi,\zeta)\hat{\rho}_S = \hat{U}(t|\xi)\hat{\rho}_S\hat{U}^\dagger(t|\xi)$ and the functional $\mathcal{Q}[\xi(t),\zeta(t)]$ is the **quasi-probability distribution**.

When the quasi-probability can be interpreted as a proper probability distribution,

$$\mathcal{Q}[\xi,\zeta]\approx\delta(\xi-\zeta)\mathcal{P}[\xi]\geqslant 0$$

then the trace reduces to stochastic average

$$\int\mathcal{D}\xi\mathcal{D}\zeta\mathcal{Q}[\xi,\zeta]\mathcal{U}(t|\xi,\zeta)\hat{\rho}_S = \int\mathcal{D}\xi\mathcal{P}[\xi]\hat{U}(t|\xi)\hat{\rho}_S\hat{U}^\dagger(t|\xi) = \overline{\hat{U}(t|\xi)\hat{\rho}_S\hat{U}^\dagger(t|\xi)}$$

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An equivalent description of quantum process ($\xi(t)$,$\zeta(t)$): an infinite family of **joint quasi-probabilities** for trajectories to pass through a given sequence of values at given times

$$q_E^{(k)}(\boldsymbol{\xi\zeta t}) = \left(\prod_{l=1}^k\sum_{\substack{n_l:e_{n_l}=\xi_l\\ m_l:e_{m_l}=\zeta_l}}\right)\delta_{n_1,m_1}\left(\prod_{l=1}^{k-1}T_{t_l-t_{l+1}}(n_l m_l|n_{l+1}m_{l+1})\right)\langle n_k|\hat{\rho}_E(t_k)|m_k\rangle$$

where the basic building block is the *propagator*

$$T_t(nm|n'm') = \mathrm{tr}_E\left(|m\rangle\langle n|e^{-it\hat{H}_E}|n'\rangle\langle m'|e^{it\hat{H}_E}\right)$$

In this language, the **sufficient condition** for the surrogate field representation to be valid is to

$$q_E^{(k)}(\boldsymbol{\xi\zeta t})\approx\delta_{\boldsymbol{\xi},\boldsymbol{\zeta}}p_E^{(k)}(\boldsymbol{\xi t})$$

for all *k*, and the **joint probability distribution** given by

$$p_E^{(k)}(\boldsymbol{\xi t}) = \left(\prod_{l=1}^k\sum_{n_l:e_{n_l}=\xi_l}\right)\left(\prod_{l=1}^{k-1}|\langle n_l|e^{-i(t_l-t_{l+1})\hat{H}_E}|n_{l+1}\rangle|^2\right)\langle n_k|\hat{\rho}_E(t_k)|n_k\rangle\geqslant 0$$

III. EXAMPLES

Examples of environment types that facilitate the surrogate field representation.

Quasi-static Coupling

The quasi-probabilities turn into proper probabilities *exactly* when

$$[\hat{E},\hat{H}_E] = 0$$

It results in the surrogate field in the form of *quasi-static noise* or static external field

$$\hat{\rho}_S(t) = \sum_\xi p(\xi)e^{-it(\hat{H}_S+\xi\hat{S})}\hat{\rho}_S\, e^{it(\hat{H}_S+\xi\hat{S})}$$

where ξ is a random variable with distribution determined by the initial state of *E*

$$p(\xi) = \sum_{n:e_n=\xi}\langle n|\hat{\rho}_E|n\rangle$$

Environment of Least Action

Let *E* be such that the **least action approximation** applies to its Feynman's path integral representation of its Schrodinger propagators

$$\langle n|e^{-i(t-t')\hat{H}_E}|n'\rangle\propto\int_{x(t')=n'}^{x(t)=n}\mathcal{D}x(\tau)e^{i\mathcal{S}_{\mathrm{cl}}[x(\tau)]}$$

where $\mathcal{S}_{\mathrm{cl}}$ is the classical *action* describing the dynamics in *E*.

If the initial density matrix is diagonal

$$\langle n|\hat{\rho}_E|n'\rangle = \delta(n-n')\rho(n)$$

i.e., it is *not* a Schrodinger's cat type of state, then due to destructive interference caused by rapidly oscillating phases, the trajectory ζ is always forced to match ξ and thus quasi-probabilites turn into proper probabilities.

Open Environment

Let *E* also be open to its environment *D* (but *S* is still only in contact with *E*) and assume that *D* induces in *E* a **markovian dynamics**, such that

$$\hat{\rho}_E(t)\approx e^{t\mathcal{L}}\hat{\rho}_E \quad\text{and}\quad e^{t\mathcal{L}} = e^{(t-s)\mathcal{L}}e^{s\mathcal{L}}$$

In such a case, the propagators transform into

$$T_t(nm|n'm') = \mathrm{tr}_E\left(|m\rangle\langle n|e^{t\mathcal{L}}|n'\rangle\langle m'|\right)$$

The quasi-probabilities reduce to when the **dynamical map** satisfies

$$e^{t\mathcal{L}}|n\rangle\langle n| = \sum_{n'}p_{n'n}(t)|n'\rangle\langle n'| \text{ and } e^{t\mathcal{L}}|n\rangle\langle m| = \sum_{n'}p_{n'm',nm}(t)|n'\rangle\langle m'|$$

Simple example: two-level *E* with markovian generator

$$\mathcal{L} = -(\gamma/2)[\hat{\sigma}_x,[\hat{\sigma}_x,\bullet]]$$

that results in the surrogate field in the form of *random telegraph noise*.

Conclusion: The variety of ways the different environmental dynamics lead to quasi-probabilities that behave as proper probability distributions indicates that **there is no one reason why the surrogate field representation is valid**.

IV. OBJECTIVITY OF SURROGATE FIELD

The quasi-probabilities are determined exclusively by *E*-side of the system--environment arrangement,

$$\left\{\hat{H}_E,\hat{E},\hat{\rho}_E\right\}\ \to\ q_E^{(k)}(\boldsymbol{\xi\zeta t})\qquad\text{so that}\qquad\hat{S}\otimes\hat{E}\ \to\ \xi(t)\hat{S}$$

Therefore, whether quasi-probabilities turn into proper probabilities is **independent of S**; in other words, if *E* facilitates the surrogate field ξ, it is **the same surrogate for any S** coupled to *E*

Conclusion: The quasi-probability-based surrogate field representation is an **universal simulator** of open system dynamics.

Conclusion: It is even possible to observe a single random trajectory of surrogate field!

Consider a sequential projective measurements of observable $\hat{E}$ performed directly on the environment. Given that *E* facilitates the surrogate field, it can be shown that the probability distribution of reading out a particular sequence of results measured at given times equals

$$\mathrm{prob}(\xi_{t_1},\xi_{t_2},\ldots,\xi_{t_k}) = p_E^{(k)}(\xi_{t_1}t_1;\xi_{t_2}t_2;\ldots,\xi_{t_k}t_k)$$

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Observation: Some noise representations are **subjective**, and thus, not all noise reps. can be classified as surrogate fields.

Consider an open system composed of dephasing qubit

$$S\ :\ \hat{H}_S = 0;\ \hat{S} = \frac{1}{2}\hat{\sigma}_z;\ \hat{\rho}_S = \frac{1}{2}(|+\rangle+|-\rangle)(\langle+|+\langle-|)$$

Within so-called Gaussian approximation the dynamics of the qubit are given by

$$\langle+|\hat{\rho}_S(t)|-\rangle\propto e^{-\frac{1}{2}\int_0^t dt_1\int_0^{t_1}dt_2\sum_{\xi_1}\sum_{\xi_2,\zeta_2}\xi_1(\xi_2+\zeta_2)q_E^{(2)}(\xi_1\xi_1t_1;\xi_2\zeta_2t_2)}$$
$$= e^{-\int_0^t dt_1\int_0^{t_1}dt_2 C(t_1,t_2)}$$

For this particular *S*, even if *E* **does not** support a surrogate field, it is always possible to construct a stochastic process $\varphi(t)$ such that its correlation function matches the form

$$\overline{\varphi(t_1)\varphi(t_2)} = C(t_1,t_2)$$

However, unless *E* facilitates surrogate field, the process $\varphi(t)$ cannot be an universal simulator, as it is incapable to account for an additional phase shift in modified open qubit system (dephasing qubit with biased coupling)

$$S'\ :\ \hat{S}' = \frac{1}{2}(\hat{1}+\hat{\sigma}_z)$$

Therefore, $\varphi(t)$ is a subjective noise--it was a good simulator for *S* but it stops working for other systems. Only proper quasi-probability-based surrogate field works for all systems.