Euler characteristic of quantum graphs and microwave networks

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INTRODUCTION

We show theoretically and experimentally that the Euler characteristic $\chi=|V|-|E|$, where $|V|$ and $|E|$ denote the number of vertices and edges of a graph, can be determined from a finite sequence of the lowest eigenenergies $\lambda_1,...,\lambda_N$ of a simple quantum graph [1]. The Euler characteristic is the most important geometrical and topological characteristic of a metric graph $\Gamma=(V, E)$ that can be used as a sensitive indicator of the fully connected graphs. Furthermore, the Euler characteristic determines the number of independent cycles in a graph

$$\beta = |E| - |V| + 1 \equiv 1 - \chi \qquad (1)$$

In the experiment quantum graphs were simulated by microwave networks [2].

QUANTUM GRAPHS AND MICROWAVE NETWORKS

Quantum graphs can be simulated by microwave networks as both systems are described by the formally equivalent equations, the one dimensional Schrödinger equation and telegrapher's equation, respectively. Knowing the spectrum of the microwave network one may recover its Euler characteristic without seeing it. The Euler characteristic χ is an integer, and for quantum graphs it is usually negative. Using k_n which are square roots of the eigenenergies λ_n the formula for the approximation function of χ is given by

$$X(t) := 2 + 2\sum_{n=1}^{\infty} \cos(k_n/2t)\left[\frac{\sin(k_n/4t)}{k_n/4t}\right]^2 \qquad (2)$$

The formula (2) is slowly convering and its aplication requires even several hundreds eigenenergies what is not available experimentally, therefore, we derieved and tested experimentally a new formula

$$X_K(t) = 2 + 8\pi^2 \sum_{n=1}^{K} \frac{\sin(k_n/t)}{(k_n/t)[(2\pi)^2 - (k_n/t)^2]} \qquad (3)$$

which gives a good approximation to the Euler characteristic for the lowest set of K resonances, if

$$t \geqslant t_0 = \frac{1}{2\ell_{min}} \qquad (4)$$

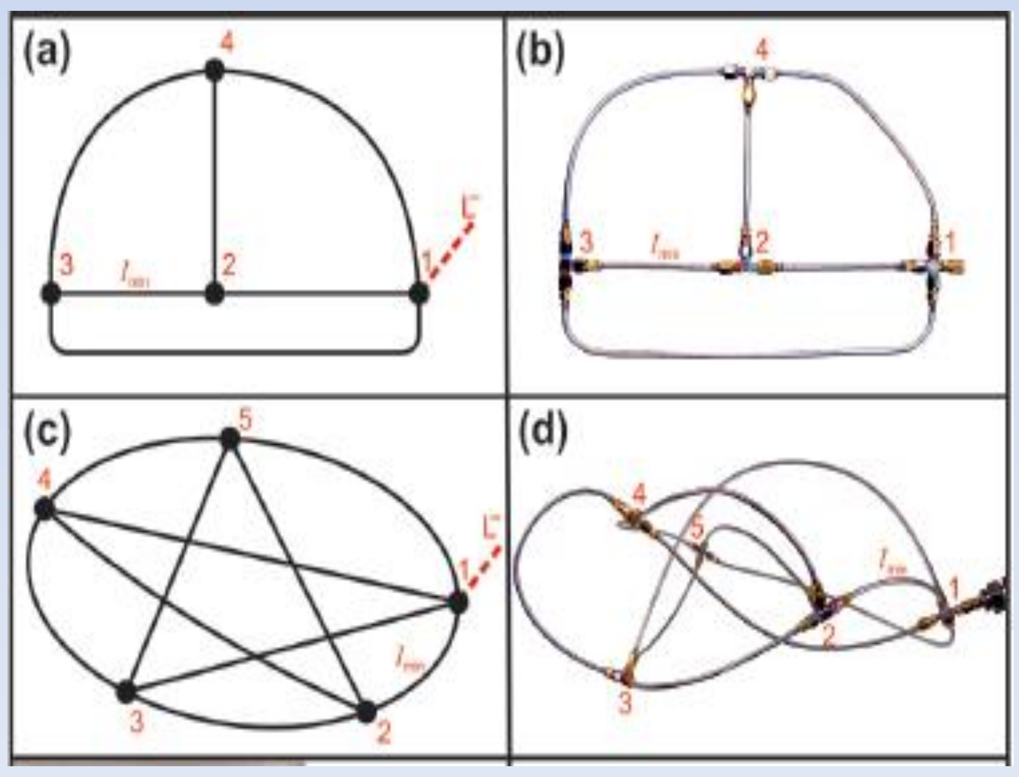

Fig.1 Panels(a)-(b) *Schemes of the planar quantum graph and simulating it microwave network of the total length* $\mathcal{L}$=1.49 m and l_{min}=0.15 *m and* $|V|$=4 *vertices and* $|E|$=6 *edges.* **Panels (c)-(d)** *The same for the non-planar graph of the length* $\mathcal{L}$=3.95 m *and* l_{min}=*0.2m with* $|V|$=5 *vertices and* $|E|$=10 *edges.*

EXPERIMENT

The resonances of microwave networks required for the evaluation of the Euler characteristic were determined from the measurements of the one-port scattering matrix $S(\nu)$ of the networks using the vector network analyzer (VNA) Agilent E8364B with the flexible microwave cable HP 85133-616. The real parts of the wave numbers k_n are directly related to the positions ν_n of the resonances $\mathrm{Re}k_n=\frac{2\pi}{c}\nu_n$. In order to avoid losing resonances we analyzed the fluctuating part of the integrated spectral counting function $N_{fl}=N(\nu_i)-N_{av}(\nu_i)$.

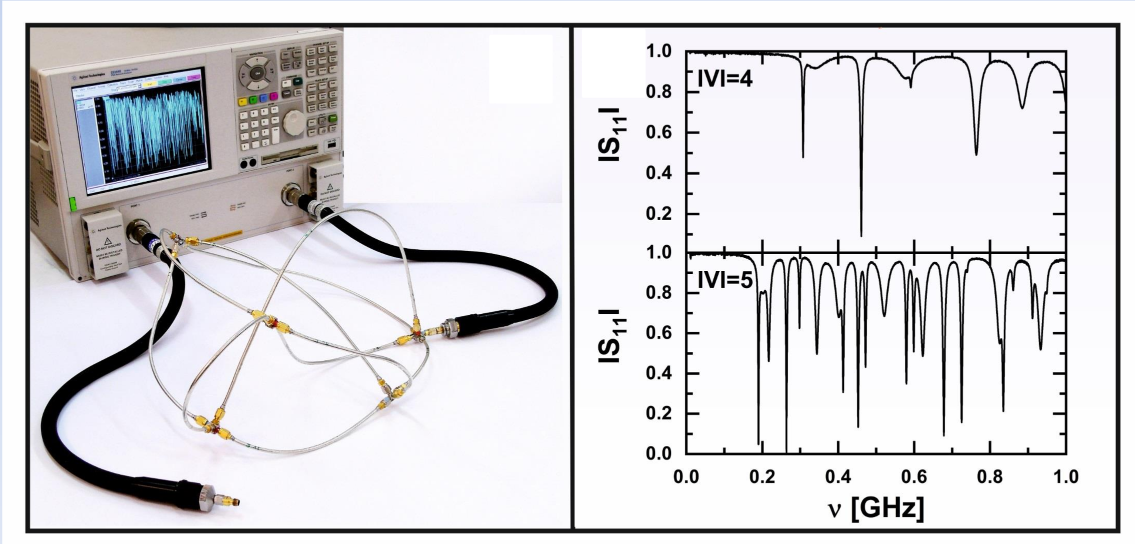

Fig.2 *The microwave network with* $|V|$=5 *vertices and* $|E|$=10 *edges connected to the VNA. Examples of the moduli of the scattering matrix* $|S_{11}(\nu)|$ *of different networks.*

EXPERIMENTAL IMPLEMENTATION

The estimate of the minimum number of resonances is given by the formula:

$$K \simeq |V| - 1 + 2\mathcal{L}t_0\left[1 - \exp\left(\frac{-\epsilon\pi}{\mathcal{L}t_0}\right)\right]^{-1/2} \qquad (5)$$

where ϵ is the estimation accuracy. Since χ is an integer, it is enough to estimate it with an accuracy better than 0.5 to compute its real value.

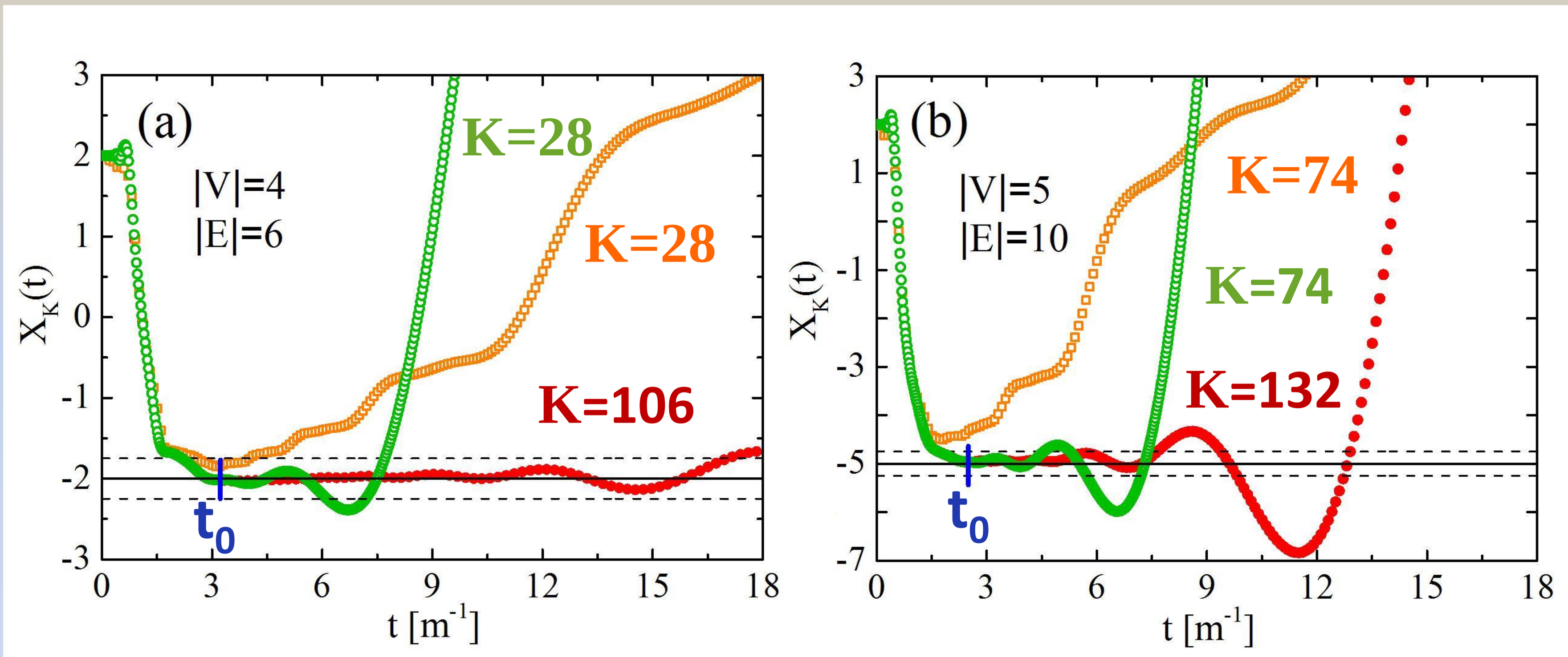

Fig.3 *The aproximation function of the Euler characteristic X(t) calculated from Eq. (3) for different K (green and red lines).* **Panel (a):** *a planar microwave network with* $|V|$=4 *vertices and* $|E|$=6 *edges*. **Panel (b):** *a non-planar microwave network with* $|V|$=5 *vertices and* $|E|$=10 *edges. The Euler characteristic* X(t) *calculated from Eq.* (2) *is presented by orange lines. The blue vertical mark indicates the value* t_0*. The black full line shows the expected values* $\chi=-2$ *for a planar graph and* $\chi=-5$ *for a non-planar graph. The* $\epsilon=\pm0.25$ *are shown by black broken lines.*

CONCLUSSIONS The Euler characteristic χ is a powerful measure of graphs' or networks' properties, including topology, complementing in an important way the inverse spectral methods that require the infinite number of eigenenergies or resonances for their application.

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